FACTA UNIVERSITATIS (NIŠ)
SER.: ELEC. ENERG. vol. 18, no. 3, December 2005, 531-545

Remarks on Systems and Differential Operators on Groups

Dedicated to Professor Milić Stojić on the occasion of his 65th birthday

Radomir S. Stanković, Miomir S. Stanković, and Claudio Moraga

Abstract: Complexity and large size of contemporary control, communication, and computer systems impose strong challenges to methods of their mathematical modelling, design, and testing. Classical approaches to their formal specification by exploiting theory of systems on groups of real numbers *R* and complex numbers *C*, often do not fulfill requirements of practice. For that reason, theory of systems on groups different from *R* and *C* has been developed. Differential operators and spectral (Fourier) analysis on groups play the same important role in such systems as in the case of systems modelled by signals defined on *R* and *C*.

This paper first briefly reviews some aspects of research in system theory on groups and then presents an extension of the notion of Gibbs differentiation to matrix-valued functions on finite non-Abelian groups.

Keywords: System theory, Differential operators, Spectral techniques, Linear Systems.

1 Introduction

Signals representing information in communication and control systems demand in processing the use of strongly time-invariant systems, and shift-invariant and rotational-invariant systems in case of speech and images, respectively. This means mathematically that the groups of real numbers *R* and complex numbers *C* are the natural domains for the definition of signals. The theory of differential calculus and

Manuscript received September2005.

R. Stanković is with Dept. of Computer Science, Faculty of Electronics, 18000 Niš, Serbia (e-mail: rstankovic@elfak.ni.ac.yu). M. Stanković is with Dept. of Mathematics, Faculty of Occupational Safety, 18 000 Niš, Serbia (e-mail: mstankovic@znrfak.znrfak.ni.ac.yu). C. Moraga is with Dept. of Computer Science, University of Dortmund, Germany (e-mail: Claudio.Moraga@uni-dortmund.de).

Fourier analysis are two very important formal methods in the study of such signals and systems.

However, in some areas of contemporary engineering practice, as for instance, in computer engineering, consideration of systems defined on various other groups may be required [1], [2]. Conversely, methods developed for such (generalized) systems may provide advantages in solving some particular tasks in classical system theory and applications. Due to that, generalizations of systems theory to systems described by signals on groups different from R or C are a subject of considerable research efforts in the last few decades. The same applies to the extensions and generalizations of the differential calculus and spectral methods on groups, that are also used in system theory on groups.

Professor M.R. Stojić contributed remarkably to these activities in different ways, by doing research and publishing [3], [4], [5], [6], [7], editorial work [8], as well as conducting and supervising MSc and PhD dissertations. This paper is intended to focus attention on this part of his academic activity.

The paper first briefly reviews research work in the area of systems on groups and related differential (Gibbs) operators in the last few decades. Then, it presents an extension of the notion of Gibbs derivatives to matrix-valued functions on finite non-Abelian groups. The paper ends with a discussion of the relationships between the Gibbs derivatives and linear shift-invariant systems on groups and their possible applications.

2 Linear Shift-invariant Systems on Groups

In this section, we briefly discuss linear convolution systems whose input and output signals are deterministic signals on groups.

Definition 1 *A scalar linear system S over a finite not necessarily Abelian group G is defined as a quadruple $(P(G), P(G), h, *)$ where the input-output relation $*$ is the convolution product on G,*

$$y = h * f, \quad f, h, y \in P(G),$$

i.e.,

$$y(\tau) = \sum_{x \in G} h(x) f(\tau \circ x^{-1}), \forall \tau \in G, \tag{1}$$

where $\circ$ is the group operation of G and $P(G)$ the space of functions $f : G \to P$, where P is a field.

An ordered pair $(f,y) \in P(G) \times P(G)$ is exactly then an input-output pair of S if f and y fulfill equation (1). The function $h \in P(G)$ is the impulse response of S.

It is easy to show that the system S is invariant against the translation of the input function. By that we mean that if y is the output to f, then $T^{\tau}y$ is the output to $T^{\tau}f$, for all $\tau \in G$. Therefore, we denote the system S as a linear translation invariant (LTI) system.

Linearity is a property very often used in providing mathematical models of physical phenomena. In that setting, the linear shift invariant systems and in particular, linear convolution systems on groups are efficiently used in mathematical modelling of real life systems. For example, in that general ground the linear time-invariant systems can be regarded as systems on the real group R. Similarly, the linear discrete-time-invariant systems are an example of systems defined on the additive group of integers Z.

Modelling of systems in terms of convolution product, naturally results in establishing the links to the Fourier analysis and exploiting advantages from the convolution theorem. These relationships extend to the systems and Fourier transforms on various groups.

It is similar to the situation with differential operators which are usually used to describe the change of the state in a system. In the case of systems on R, the Newton-Leibniz derivative is used. For systems on groups, the Gibbs derivatives are a good candidate, especially since they can be alternatively defined by themselves in terms of the convolution product on the group.

Note that from the system theory point of view, S is a linear input/output system whose input and output are defined over an arbitrary group G. By using different groups, various systems studied by several authors can be obtained.

It is pointed out in [9] that the notion of dyadic invariant linear systems has been introduced in [10], if G is the dyadic group. Theory of linear systems on dyadic groups has been contributed by the work of Harmuth in 1964, and supported by the mathematical background theory provided by Polyak and Shreider [11], [12].

The dyadic systems have been intensively studied by Pichler in a series of papers and by several other authors; see [13] for a bibliography up to 1989. For more recent result, we refer to [14], [15], [16].

The systems where input and output signals are modelled by functions mapping the infinite cyclic group of integers into Galois fields of order 2^q, $q \in N$, $GF(2^q)$ have been considered by Tsypkin and Faradzhev [17]. A generalization of the concept was given in [18] where it was shown that both cyclic and dyadic convolution systems on finite groups can be regarded as special classes of permutation-invariant systems. A special class of discrete-time systems with variable structure over a finite interval $[0, g-1]$ has been considered in [19]. In a more general setting, systems over locally compact Abelian groups were considered by Falb and Friedman

[20]. Some aspects of the theory were extended also to non-Abelian groups by Karpovsky and Trachtenberg [1], [2].

Recall that systems over finite groups can be regarded as a special class of digital filters [21], [22], [23] or a special class of discrete-time systems with variable structure [19] over a finite interval $[0, g-1]$, see also [24].

Regarding the applications of these systems note that they can be used as models of both information channels (for example to represent an encoder, a digital filter, or a Wiener filter if K is the field of complex numbers and u is a stochastic signal), and computation channels if K is a finite field. For example, different criteria for the approximation of linear time-invariant systems by linear convolution systems on groups were discussed in [24].

If G is the group Z_{p^n} we obtain the systems studied in [25], [26], [27], and [28] called somewhere as p-adic systems.

3 Gibbs Derivatives and Linear Systems

Differential operators are used in linear systems theory to describe the change of state of a system. The systems on R described by linear differential equations with constant coefficients in terms of Newton-Leibniz derivative are probably the most familiar example. However, the group theoretic models of systems and Gibbs derivatives on groups, in particular on the dyadic groups C_2^n and groups Cp^n, where C_2 and C_p are cyclic groups of orders 2 and p, respectively, have attained some considerable attention [15],[26], [28],[29], [30], [31], [32], [33], [34], [35].

The relationship between linear convolution systems on locally compact Abelian and finite non-Abelian groups discussed above can be considered and summarized in a general setting as follows.

In a general ground the Gibbs differentiator of order k of a function $f \in P(G)$, which we denote by $D^k f$, is considered as the linear operator in $P(G)$ satisfying the relationship [36]

$$(F(D^k f))(w) = \varphi(w,k)(F(f))(w), \tag{2}$$

where F denotes the Fourier transform operator in $P(G)$.

In most cases $\varphi(w,k) = w^k$, but in some cases a scaling factor should be applied, while in a few particular cases the function φ differs and is related to the order of group G. For example, in the case of the extended Butzer-Wagner dyadic derivative [37] $\varphi(w,k) = (w^*(w))^k$, where

$$w^*(w) = \sum_{i=0}^{\infty} (-1)^i w_i 2^i,$$

w_i being the coefficients in the dyadic expansion of $w \in P$. In the case of Gibbs derivatives on Vilenkin groups [38], [39], [40], the function $\varphi(w,k)$ is a function from the so-called symbol class $S^m_{\rho,\sigma}$ [38] defined as $\varphi(w,k) = \langle k \rangle^m, m \geq 0$, where $\langle x \rangle = \max\{1, \|x\|\}$.

In attempting to determine a relationship between Gibbs derivatives and linear convolution systems we want to point out that

1. Thanks to the relation (2) and the convolution theorem in the Fourier analysis on groups, the Gibbs differentiator of order k can be considered as a convolution operator and, therefore, can be identified with a linear convolution system whose impulse response function h is given in the transform domain by $(F(h))(w) = \varphi(w,k)$. For example, in the case of the Gibbs derivative on finite not necessarily Abelian groups, as well as in the case of dyadic and groups Cp^n, $\varphi(w,k) = w^k$ by definition and, therefore, h is the k-th Gibbs derivative of the δ-function defined as $\delta(x) = 1$ for x equals the unit element of G, and $\delta(x) = 0$ otherwise.
2. A considerable class of linear systems on groups can be described by Gibbs differential equations in a way resembling the use of classical differential equations with constant coefficients in the linear system theory on the real group R. In other words, a linear Gibbs differential on a group is defined as a polynomial in the Gibbs differentiator with real coefficients. The linear Gibbs differential operator form a subset of the group convolution operators realized by the corresponding subset of the group convolution systems.

As we noted above such linear systems over dyadic groups were discussed in [41], [42], and for the infinite dyadic groups in [32]. A generalization to finite non-Abelian groups was given in [43], see also [44] and for p-adic systems with stochastic signals in [15].

Note that the use of systems modelled by Gibbs differential equations in the processing of two-dimensional signals was suggested in [45], [46].

4 Gibbs Derivatives for Matrix-Valued Functions

In this section, we extend the notion of Gibbs differential operators to matrix-valued functions on finite non-Abelian groups. Linear-shift invariant systems modelled by matrix-valued functions have already been considered, see for example, [1], and the present interest in them may be related to the reconfigurable networks for different calculation tasks over the same structure.

Denote by $f \in P(G)$ the space of functions $f : G \to P$, where G is a finite group of order $|G| = g$ and P a field that may be the complex field C or a finite (Galois) field. A function $f \in P(G)$ is conveniently represented as a vector of its values at all the points of G, as $\mathbf{F} = [f(0), \ldots, f(g-1)]^T$.

Definition 2 *(Matrix-valued functions)*
A function f defined on a group G and taking values in a set of $(a \times b)$ matrices $M_{a,b}$ over a field P is called a matrix-valued function on G.

The space of all matrix-valued function on G over P is denoted by $P_{a,b}(G)$. A function $f \in P_{a,b}(G)$ is represented by a vector $[\mathbf{F}] = [\mathbf{f}(0), \ldots, \mathbf{f}(g-1)]^T$, where $\mathbf{f}(i)$ are $(a \times b)$ matrices over P.

Definition 3 *The Gibbs derivative D for functions in $P_{a,b}(G)$ is defined as*

$$\mathbf{D}_f = g^{-1}[\mathbf{R}] \cdot ([\mathbf{G}] \odot [\mathbf{R}]^{-1}),$$

where $[\mathbf{R}]$ is the matrix of unitary irreducible representations of G over P, i.e., $[\mathbf{R}] = [\mathbf{a}_{ij}]$ with $\mathbf{a}_{ij} = \mathbf{R}_j(i)$, $i \in \{0, 1, \ldots, g-1\}$, $j \in \{0, 1, \ldots, K-1\}$, $\mathbf{G}$ is a column vector of K elements defined as $\mathbf{G} = [k\mathbf{I}_{r_{w_k}}]$, $k \in \{0, 1, \ldots, K-1\}$, and $\mathbf{I}_{r_{w_i}}$ is the $(r_{w_i} \times r_{w_i})$ identity matrix with r_{w_i} the order of $R_{r_{w_i}}$, and $[\mathbf{R}]^{-1} = [\mathbf{b}_{sq}]$ with $\mathbf{b}_{sq} = r_s\mathbf{R}^{-1}(q)$, $s \in \{0, 1, \ldots, K-1\}$, $q \in \{0, 1, \ldots, g-1\}$. In this relation, $\cdot$ stands for the generalized matrix multiplication permitting to multiply matrices whose entries are matrices, and $\odot$ is this multiplication performed componentwise.

This definition will be illustrated by the example of the Gibbs derivative for matrix-valued function on the Quaternion group Q_2.

Example 1 *The unitary irreducible representation of the quaternion group Q_2 are given in Table 1 where*

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Therefore, Table 1 defines the matrix $[\mathbf{Q}]$ of unitary irreducible representations of Q_2. The inverse matrix is

$$[\mathbf{Q}]^{-1} = \frac{1}{8} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 2\mathbf{I} & 2i\mathbf{B} & -2\mathbf{I} & 2i\mathbf{A} & 2\mathbf{E} & 2i\mathbf{D} & 2\mathbf{C} & -2i\mathbf{D} \end{bmatrix}.$$

Notice that $[\mathbf{Q}]^{-1}$ *defines the Fourier transform on* Q_2.

In analytical form, the direct and inverse Fourier transform of $f \in P_{2,2}(Q_2)$ *are defined respectively by*

$$\begin{aligned} \mathbf{S}_f(w) &= (S_f^{(i,j)}(w)) = r_w g^{-1} \sum_{u=0}^{7} f^{(i,j)}(u)\mathbf{R}_w(u^{-1}), \\ \mathbf{f}(x) &= (f^{(i,j)}(x)) = \sum_{w=0}^{4} Tr(\mathbf{S}_f^{(i,j)}(w)\mathbf{R}_w(x)). \end{aligned}$$

The matrix $[\mathbf{G}]$ *in* $P(Q_2)$ *is given by*

$$[\mathbf{G}] = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \end{bmatrix}.$$

Therefore, the Gibbs derivative for a function $f \in P_{2,2}(Q_2)$ *is defined as*

$$\mathbf{D}_f = \frac{1}{8}[\mathbf{Q}] \cdot ([\mathbf{G}] \odot [\mathbf{Q}]^{-1}),$$

and, after simple calculations, where when multiplying matrix-valued entries of a matrix, the trace of the resulting matrix-valued entry is taken,

$$\mathbf{D}_f = \frac{1}{8} \begin{bmatrix} 22 & -4 & -10 & -4 & -2 & 0 & -2 & 0 \\ -4 & 22 & -4 & -10 & 0 & -2 & 0 & -2 \\ -10 & -4 & 22 & -4 & -2 & 0 & -2 & 0 \\ -4 & -10 & -4 & 22 & 0 & -2 & 0 & -2 \\ -2 & 0 & -2 & 0 & 22 & -4 & -10 & -4 \\ 0 & -2 & 0 & -2 & -4 & 22 & -4 & -10 \\ -2 & 0 & -2 & 0 & -10 & -4 & 22 & -4 \\ 0 & -2 & 0 & -2 & -4 & -10 & -4 & 22 \end{bmatrix}$$

Notice that the matrix $\mathbf{D}_f$ *has the structure of the convolution matrix on* Q_2. *Therefore, the Gibbs derivative in* $P_{2,2}(Q_2)$ *can be defined as*

$$\mathbf{D}_f = \mathbf{f} * q,$$

where q *is defined by the vector* $\mathbf{q}$ *of it values as*

$$\mathbf{q} = \frac{1}{8}[22, -4, -10, -4, -2, 0, -2, 0]^T.$$

and $*$ denotes the convolution product as defined in (1) applied to matrix-valued functions.

For the function $f \in P_{2,2}(Q_2)$ given by the vector of its values

$$[\mathbf{F}] = \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]^T,$$

the Gibbs derivative is

$$\begin{aligned}[\mathbf{D}_f] &= \frac{1}{8}\left[\begin{bmatrix} -6 & 6 \\ 0 & -6 \end{bmatrix}, \begin{bmatrix} 20 & -20 \\ 0 & 20 \end{bmatrix}, \begin{bmatrix} -6 & 6 \\ 0 & -6 \end{bmatrix}, \begin{bmatrix} -12 & 12 \\ 0 & -12 \end{bmatrix},\right. \\ &\left.\begin{bmatrix} -14 & 14 \\ 0 & -14 \end{bmatrix}, \begin{bmatrix} -16 & 16 \\ 0 & -16 \end{bmatrix}, \begin{bmatrix} 18 & -18 \\ 0 & 18 \end{bmatrix}, \begin{bmatrix} -16 & 16 \\ 0 & -16 \end{bmatrix}\right]^T \\ &= \mathbf{r} \odot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\end{aligned}$$

where $\mathbf{r} = \frac{1}{8}[-6, 20, -6, -12, -14, -16, 18, -16]^T$.

The Fourier spectrum of the considered function $\mathbf{f}$ *is given by*

$$[\mathbf{S}_f] = [\mathbf{S}_f(0), \mathbf{S}_f(1), \mathbf{S}_f(2), \mathbf{S}_f(3), \mathbf{S}_f(4)]^T,$$

where the Fourier coefficients are

$$\begin{aligned}\mathbf{S}_f(0) &= \frac{1}{8}\begin{bmatrix} 3 & 5 \\ 8 & 3 \end{bmatrix}, \\ \mathbf{S}_f(1) &= \mathbf{S}_f(2) = \mathbf{S}_f(3) = \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},\end{aligned}$$

and

$$\mathbf{S}_f(4) = \frac{1}{4}\begin{bmatrix} \begin{bmatrix} -i & -1+i \\ 1+i & i \end{bmatrix} & \begin{bmatrix} i & 1-i \\ -1-i & -i \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -i & -1+i \\ 1+i & i \end{bmatrix} \end{bmatrix}.$$

The Fourier spectrum of D_f is given by

$$[\mathbf{S}_{D_f}] = [\mathbf{S}_{D_f}(0), \mathbf{S}_{D_f}(1), \mathbf{S}_{D_f}(2), \mathbf{S}_{D_f}(3), \mathbf{S}_{D_f}(4)]^T,$$

where the Fourier coefficients are

$$\begin{aligned}
\mathbf{S}_{D_f}(0) &= \frac{1}{8}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
\mathbf{S}_{D_f}(1) &= \frac{1}{8}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \\
\mathbf{S}_{D_f}(2) &= \frac{1}{8}\begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, \\
\mathbf{S}_{D_f}(3) &= \frac{1}{8}\begin{bmatrix} -3 & 3 \\ 0 & -3 \end{bmatrix},
\end{aligned}$$

and

$$\mathbf{S}_{D_f}(4) = \begin{bmatrix} \begin{bmatrix} -i & -1+i \\ 1+i & i \end{bmatrix} & \begin{bmatrix} i & 1-i \\ -1-i & -i \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} -i & -1+i \\ 1+i & i \end{bmatrix} \end{bmatrix}.$$

Therefore, it is easy to verify that the relation (2) expressing main feature of Gibbs derivatives is true, i.e., in this case

$$[\mathbf{S}_{\mathbf{D}_f} = [\mathbf{G}] \odot [\mathbf{S}_f].$$

This Gibbs differential operator expresses all the properties of Gibbs differentiators on groups [36], and fast calculation algorithms for it can be derived in a way similar to that used in [47].

The example above illustrates the principle of defining the Gibbs derivatives for matrix-valued functions on finite non-Abelian groups for the case of the quaternion group Q_2 as the domain, and the complex field P for the range of signals processed. Extensions and generalizations to signals described by functions defined on other groups and direct products of groups, and taking values in different fields admitting the existence of a Fourier transform for the group considered, are straightforward.

5 Closing Remarks

The dyadic derivative is a differential operator especially adapted to functions having many jumps and possessing just a few and short intervals of constancy. Even functions having a denumerable set of discontinuities like the well-known Dirichlet

Table 1. Irreducible unitary representations of Q_2 .

x	$\mathbf{R}_0$	$\mathbf{R}_1$	$\mathbf{R}_2$	$\mathbf{R}_3$	$\mathbf{R}_4$
0	1	1	1	1	$\mathbf{I}$
1	1	1	-1	-1	$i\mathbf{A}$
2	1	1	1	1	$-\mathbf{I}$
3	1	1	-1	-1	$i\mathbf{B}$
4	1	-1	1	-1	$\mathbf{C}$
5	1	-1	-1	1	$-i\mathbf{D}$
6	1	-1	1	-1	$\mathbf{E}$
7	1	-1	-1	1	$i\mathbf{D}$
	$r_0 = 1$	$r_1 = 1$	$r_2 = 1$	$r_3 = 1$	$r_4 = 2$

function can be dyadically differentiable on $[0,1]$. In the case of finite groups, the Gibbs derivatives also provide a mean to differentiate functions on those groups. In one word, through the family of Gibbs differentiators, the advantage of the use of differential calculus extends to the theory of systems whose input/output signals are piecewise constant, or discrete functions.

In order to point out some possible advantages of linear systems on groups modelled by the Gibbs differential equations, recall that the use of Fourier analysis in linear systems theory is based upon the convolution theorem and the relationship between the Newton-Leibniz derivative. Thanks to the first property, the Fourier transform maps the convolution into ordinary multiplication, while the second permits the translation of differential equations into the algebraic ones. As in many other areas, the application of Fourier analysis in linear systems theory is further supported by the existence of the fast Fourier transform, FFT, and related algorithms for efficient calculation of Fourier coefficients and some other parameters useful in practical applications.

The Gibbs derivatives possesses most of the useful properties of Newton-Leibniz derivative, except the product rule and, therefore, their role in the theory of linear systems on groups can be compared to that of Newton-Leibniz derivative in classical linear systems theory on *R*. At the same time, the Gibbs derivatives are efficiently characterized by the Fourier coefficients on groups.

The matrices representing Gibbs derivatives are Kronecker product representable in the case of finite decomposable groups, and, therefore, fast algorithms for the calculation of the values of Gibbs derivatives on these groups can be defined.

It may be said that the Gibbs differentiation shares some of very useful properties of both Fourier analysis and differential calculus.

Thanks to these properties the Gibbs derivatives could be very promising for the use in the theory of linear systems on groups. Some recent results and extensions of the theory are given in [14], [16].

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Personal remarks from Radomir S. Stanković

Postscript on Teaching, Research, and Friendly Advising

It has been quite long after I was introduced by Miomir to Prof. Milić R. Stojić. Although we both were already graduated and liberated of usual student's responsibilities, we continued to attend lectures of Prof. Stojić, being interested not just in the subject, but also mainly due to an outstanding way of lecturing. I clearly remember an answer that we got when asking him about teaching, *Each lecture is a performance and, as professionals, you should always be well prepared and much convincing. Students should have the feeling that the science you are explaining has been created right now and in front of them*.

Lectures were scheduled on Friday afternoons, since Prof. Stojić devotedly accepted to sacrifice his weekends travelling from Belgrade to Niš, to help development of this branch of science at a young, for a University, Faculty of Electronics. This appeared a very convenient schedule that we liked, since after the regular part, lectures usually continued deep in the evening and often night, by slowly changing the subject from scientific and research topics, to history, mostly Serbian, through culture and art, to the way of writing and literature, first mainly English, and then, almost as seeking for an equilibrium, as it should be in a good stable automatic system, with a twist to the Russian literature, with suddenly few sentences in perfect Russian.

We used to ask many questions on various subjects. That was a very good way to learn. It was not always easy to recognize whether something he said has been a straightforward strong comment, a bitter irony, or just an innocent joke. For many comments, we have to think twice: shall we take them straightforwardly or in the completely opposite sense, and we easily realized that the justification for the nickname Era given to Prof. Stojić, has much deeper than geographical roots. To decide, it was necessary to be a good listener, careful observer of all the circumstances, and continuously keep a global overview of all aspects of the current and past discussions.

These premises for participation in the conversation, actually were a very good introduction and training for accepting the same policy in a research work. Research tasks, often quite sophisticated, were given in a simple way, during the conversation, almost unnoticeably, and guidance towards solutions formulated through questions. We have often had the feeling that these are problems we should already ask ourselves and definitely must try to learn more.

In such circumstances, it was not surprising that we soon reached the period of learning first steps in scientific publishing. Friendly advising and the guidance by Prof. Stojić that we had the privilege to follow, have been very important and equally interesting due, for instance, among other things, also to many words borrowed and customized from German that had been used in printing, which are now replaced by a more modern computer-related terminology.

The same way of study and research, always with strong, but friendly, criticism, advising, and support, continued for many years through participation at projects conducted by Prof. Stojić, who always provided room for research also for topics such as discussed in the above paper, and the relationships established helped us quite much not to give up, resists, and continue research in spite of all the circumstances that Serbia went through.

Similar research interests, high professionalism, and first of all, willingness to help youngsters to learn, has been a natural basis that, from the first meeting in Serbia, when that finally become possible few years ago, Prof. Moraga and Prof. Stojić, have immediately found a high common understanding and jointed efforts to further help the Faculty of Electronics in providing a better millieu for research and study.

We hope that joint work and friendship with Prof. Stojić will be long lasting and as enjoyable as always in the future.