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Comparative Study of Zero Effects in Pole-Placement Control System Design Via the Shift and Delta Transforms

Dedicated to Professor Milić Stojić on the occasion of his 65th birthday

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Abstract: This paper deals with the special replacement of the shift operator and its associated $z-$ transform by delta operator and $\Delta-$ transform, respectively. The aim of the paper is to clarify the role of zeros of discretized linear single input single output continuous-time systems modelled by shift and delta operators. In particular, the effect of zero dynamics on the control system design based on classical pole-zero assignment in the case of both operators is considered. The analysis is illustrated by simulation results.

Keywords: Discrete-time control, delta operator, delta domain control, pole placement.

1 Introduction

Traditional digital signal processing and control algorithms, developed during the past five decades, employ the technique of the shift operator to represent the dynamics of sampled data systems. However, the shift operator does not overcome the gap between the high sampling rates of widely available digital-signal-processing chips, and relative slow dynamics of the continuous-time processes. In such situations of processing and control data, often in real time at very high speeds, a more suitable mathematical operator is necessary.

Middleton and Goodwin [1],[2] developed a unified description of continuous-time and discrete-time systems. It allows continuous-time results to be obtained as a particular special case of discrete-time ones, by setting the sampling period to zero. The new approach is based on the introduction of the so-called delta ($\delta-$) operator

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as an alternative to the shift operator. In recent years, the $\delta-$operator methodology has been widely accepted as an effective tool, well matched to modern control system design procedures, for system modelling and identification, computer control, high-speed digital signal processing, and fast sampled data representation. The delta operator establishes a special rapprochement between analog and discrete dynamic models and allows for investigating the asymptotic behaviour of discrete-time models as the sampling period converges to zero.

Numerous advantages, for using the delta operator rather than the shift operator, have been listed and studied still by Middleton and Goodwin [3]. Among other things, they pointed to the advantages of the delta operator by optimum state-estimation, system identification and time-series modelling, as well as by control system design. In the shift form, as the sampling rate increases, the poles and zeros cluster around the point (1, j0) in the $z-$ plane and the solution algorithms are better conditioned in delta than in shift form. From then on, the delta operator became more attractive, and interesting links between continuous-time and discrete-time signals and systems analysis had been established [3], [4], [5], [6]. On the other hand, some limitations of the $\delta-$domain setting have been also reported, e.g. it is a common opinion that the relevant $\delta-$based computations become more complicated [7].

Some researchers in control theory have sought to solve the problem of unstable sampling zeros at fast sampling rates. They propose the use of the delta operator to overcome the unstable as well as the nonminimum sampling zero problem in one relative simple way [8],[9]. Finding efficient stability tests for delta operator formulated discrete-time systems was another topic of interest in the area of high-speed signal processing and control [10], [11].

It is well known, that the stability of a digital control system may be lost due to the finite word length effects at practical implementation of digital control or filter algorithms. Some filter realizations (direct form, for example) are inherently high sensitive to small coefficient changes, and thus the coefficient rounding errors may cause significant errors in a filter implementation. This problem is more serious when fast sampling is used. Delta operator, however, provides superior rounding error performance and coefficient sensitivity properties [12], [2],[13].

The $\delta-$ operator methodology promoted by Middleton and Goodwin [1] had been tested in a teaching environment at the University of Newcastle, Australia for years. Moreover, this encouraged professor Middleton to write the software and documentation for the Delta Toolbox [14] which can be downloaded from his personal site.

The layout of the paper is as follows: in Section 2 a brief review of the delta operator is given; Section 3 addresses to the structure of the zeros of discrete-time models; Section 4 contains an illustrative example; Section 5 examines pole placement design based on both shift and delta object models.

2 Preliminary

Let us define the time domain $t \in \Omega(T)$ as follows

$$\Omega(T) = \begin{cases} \Re^+ \cup \{0\}, & \text{for} \quad T = 0 \\ \{0, T, 2T, \ \dots \} & \text{for} \quad T \neq 0, \end{cases} \tag{1}$$

where T denotes the sampling period (for continuous time $T = 0$) and $\Re^+$ is the set of positive real numbers.

The wellknown **forward shift operator**, q , may be defined for $T \neq 0$ as

$$qx(\cdot) \overset{\text{def}}{=} y(\cdot) \tag{2}$$

where $y(t) \overset{\text{def}}{=} x(t+T)$ for all $t \in \Omega$

The **delta operator** is defined for the different values of the sampling period as

$$\delta(T) \overset{\text{def}}{=} \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(\cdot), & \text{for} \quad T = 0 \\ \dfrac{q-1}{T}, & \text{for} \quad T \neq 0, \end{cases} \tag{3}$$

and

$$\delta x(t) \overset{\text{def}}{=} \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}x(t), & \text{for} \quad T = 0 \\ \dfrac{x(t+T) - x(t)}{T}, & \text{for} \quad T \neq 0. \end{cases} \tag{4}$$

Recall, that δ, as given in (2), is Euler's approximation of derivative, known in the numerical analyze. Thus, the delta transform represents an alternative discrete transform (see Fig. 1, [11]). Based on the delta operator, the unified calculus in the case of continuous- as well as discrete-time systems can be used. Moreover, if we regard the sampling period T as a perturbation parameter, the δ - system model can be considered as a regular perturbation of continuous-time system model [1].

Consider the linear continuous-time single input single output system given by state and output equations

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}_c \boldsymbol{x}(t) + \boldsymbol{b}_c u(t) \\ y(t) &= \boldsymbol{c}\boldsymbol{x}(t) \ , \quad \boldsymbol{x} \in \Re^n, \quad u, y \in \Re \ , \end{aligned} \tag{5}$$

or the corresponding transfer function

$$G_c(s) = \boldsymbol{c}\,(s\boldsymbol{I} - \boldsymbol{A}_c)^{-1} \boldsymbol{b}_c \ . \tag{6}$$

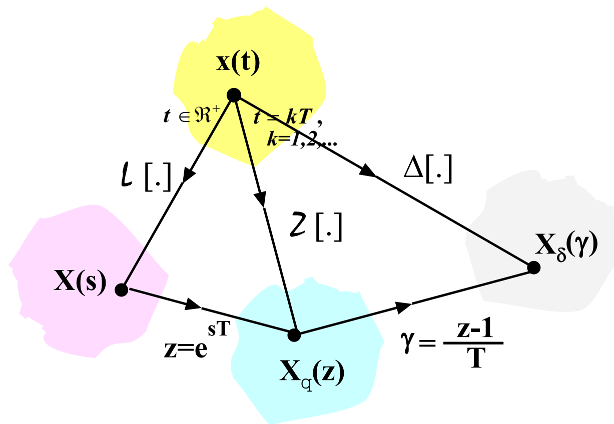


Fig. 1. Mutual relationships between the time- and complex-domains

For the sake of clarity, it is suitable to introduce the realization set S_c noted as

$$S_c \overset{\text{def}}{=} \left\{ (\boldsymbol{A}_c, \boldsymbol{b}_c, \boldsymbol{c}) : \; G_c(s) = \boldsymbol{c}\,(s\boldsymbol{I} - \boldsymbol{A}_c)^{-1} \boldsymbol{b}_c \right\} \quad . \tag{7}$$

The sampled zero-order hold representation of system (7) can be written in shift form as $q-$ model given by

$$\begin{aligned} q\boldsymbol{x}(t) &= \boldsymbol{A}_q \boldsymbol{x}(t) + \boldsymbol{b}_q u(t) \\ y(t) &= \boldsymbol{c}\boldsymbol{x}(t) \quad , \end{aligned} \tag{8}$$

as well as in delta form as $\delta-$ model given by

$$\begin{aligned} \delta\boldsymbol{x}(t) &= \boldsymbol{A}_\delta \boldsymbol{x}(t) + \boldsymbol{b}_\delta u(t) \\ y(t) &= \boldsymbol{c}\boldsymbol{x}(t) \quad . \end{aligned} \tag{9}$$

Therefore the corresponding discrete-time realization sets are

$$S_q \overset{\text{def}}{=} \left\{ (\boldsymbol{A}_q, \boldsymbol{b}_q, \boldsymbol{c}) : \; G_q(z) = \boldsymbol{c}\,(z\boldsymbol{I} - \boldsymbol{A}_q)^{-1} \boldsymbol{b}_q \right\} \quad , \tag{10}$$

and

$$S_\delta \overset{\text{def}}{=} \left\{ \left(\boldsymbol{A}_\delta, \boldsymbol{b}_\delta, \boldsymbol{c}\right) : \; G_\delta(\gamma) = \boldsymbol{c}\left(\gamma\boldsymbol{I} - \boldsymbol{A}_\delta\right)^{-1} \boldsymbol{b}_\delta \right\} \quad . \tag{11}$$

One can show that

$$\boldsymbol{A}_q = \boldsymbol{I} + \boldsymbol{A}_c T\,\Psi(\boldsymbol{A}_c, T) \; , \; \boldsymbol{b}_q = T\,\Psi(\boldsymbol{A}_c, T)\,\boldsymbol{b}_c \; , \tag{12}$$

and

$$\boldsymbol{A}_\delta = \boldsymbol{A}_c \Psi(\boldsymbol{A}_c, T) \; , \; \boldsymbol{b}_\delta = \Psi(\boldsymbol{A}_c, T)\,\boldsymbol{b}_c \; , \tag{13}$$

where

$$\Psi(\boldsymbol{A}_c, T) = \boldsymbol{I} + \frac{\boldsymbol{A}_c T}{2!} + \frac{\boldsymbol{A}_c^2 T^2}{3!} + \cdots \quad . \tag{14}$$

Notice that

$$\lim_{T\to 0} \boldsymbol{A}_q = \boldsymbol{I}\ ,\ \lim_{T\to 0} \boldsymbol{b}_q = \boldsymbol{0}\ ,\ \lim_{T\to 0} G_q(z) = 0\ , \tag{15}$$

and

$$\lim_{T\to 0} \boldsymbol{A}_\delta = \boldsymbol{A}_c\ ,\ \lim_{T\to 0} \boldsymbol{b}_\delta = \boldsymbol{b}_c\ ,\ \lim_{T\to 0} G_\delta(\gamma) = G_c(\gamma)\ . \tag{16}$$

Equations (15) and (16) show that, as the sampling rate increases, the state-space model in delta domain converges to the underlying continuous-time model, whereas the limit of the shift domain model is uninformative.

3 Zeros of Discrete-Time Models

Let the continuous-time transfer function $G_c(s)$ be a real rational function. Assume that the difference between the number of poles and the number of zeros is the system relative degree, i.e., $l = n - m$.Recall, that it is not possible to give a simple formula for the mapping of continuous-time zeros to the discrete-time ones. However, it is shown [15] that, as the sampling period $T \to 0$, m zeros of $G_q(z)$ go to 1 as $\exp(s_i T)$, where $s_i,\ i = 1, \ldots, m$ are zeros of $G_c(s)$; the remaining $l-1$ zeros of $G_q(z)$ converge to the zeros of $B_{n-m}(z)$, where $B_k(z)$ is the polynomial

$$B_k(z) = b_1^k z^{k-1} + b_2^k z^{k-2} + \cdots + b_k^k \tag{17}$$

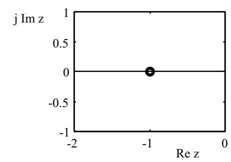
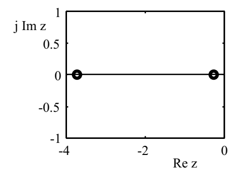
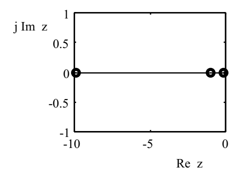
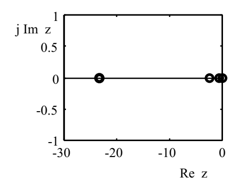
and

$$b_i^k = \sum_{j=1}^{i} (-1)^{i-j} j^k \begin{pmatrix} k+1 \\ i-j \end{pmatrix}\ ,\ \ i = 1, \ldots, k \tag{18}$$

The first five polynomials $B_k(z)$ and their zeros are given in Table 1. Notice that the zeros of the shift operator model introduced by sampling (sampling zeros) depend on sampling period T , and may become unstable in the fast sampling limit. It implies that some control methods, that require the plant to be minimum-phase, cannot be used when the sampling is very fast. To avoid the problems with the nonminimum-phase discrete-time models, the use of delta operator parameterization is recommended.

It is shown that, under sampling of a linear continuous-time single-input single output system (5) of relative degree $l \geq 2$, the zero dynamics of the resulting discrete-time δ - model (9) is singularly perturbed [1, 8] . Note, that some results of Tesfaye and Tomizuka [8] were rectified in the comments of Weller [9]. Then system (9) has $n-1$ zeros which, according to their asymptotic behavior as $T \to 0$, belong to two grups, as follows

Table 1. The first five polynomials B_k and their zeros.

k	B_k	Zeros Location
1	$B_1(z) = 1$	◊◊◊◊◊
2	$B_2(z) = z + 1$	
3	$B_3(z) = z^2 + 4z + 1$	
4	$B_4(z) = z^3 + 11z^2 + 11z + 1$	
5	$B_5(z) = z^4 + 26z^3 + 66z^2 + 26z + 1$	

1. The $l-1$ sampling zeros converge asymptoticaly to the set $T^{-1}\lambda(W_1)$, where $\lambda(W_1)$ denotes the set of eigenvalues of the matrix

$$W_1 = \begin{bmatrix} -\frac{\alpha_1(l)}{\alpha_0(l)} & 1 & 0 & \cdots & 0 \\ -\frac{\alpha_2(l)}{\alpha_0(l)} & 0 & 1 & & 0 \\ \vdots & & & \ddots & \\ -\frac{\alpha_{l-2}(l)}{\alpha_0(l)} & & & & 1 \\ -\frac{\alpha_{l-1}(l)}{\alpha_0(l)} & 0 & \cdots & \cdots & 0 \end{bmatrix} \in \Re^{(l-1)\times(l-1)}, \tag{19}$$

and

$$\alpha_i(l) = \boldsymbol{c}\boldsymbol{A}_c^i \left\{ \sum_{j=0}^{i+1} \binom{i+1}{j} \left[(T\boldsymbol{b}_1^*)^j \right]_{l-i-1} \right\} \boldsymbol{b}_c, \quad i = 0, 1, \ldots, l-1, \tag{20}$$

where $\left[(T\boldsymbol{b}_1^*)^j\right]_k$ denotes the coefficient of T^k in the expansion of $(Tb_1^*)^j$ and $\boldsymbol{b}_1^* = \Psi(\mathbf{A}_c, T) - \mathbf{I}$;

2. The remaining $n-l$ zeros tend to the finite zeros of the continuous-time system (5).

4 Illustrative Example

Consider a continuous-time system with the following transfer function

$$G_c(s) = \frac{(s^2+s+1)(s^2+0.4s+4)}{(s+1)(s^2+4)(s^2+9)(s^2+16)} \ . \tag{21}$$

The relative degree of the system is three and transfer function pole-zero location is shown in Fig. 2. Figures 3(a) and 3(b) visualize the influence of the sampling period T on the location of the poles and zeros for the $q-$ and $\delta-$ model, respectively.

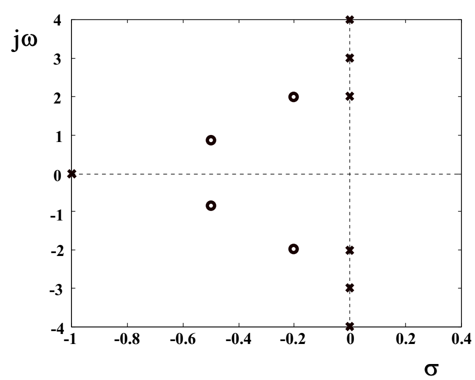


Fig. 2. Pole-zero location of transfer function (21)

For the shift operator model we notice that two zeros, introduced by sampling, converge to the zeros of the polynomial $B_3(z)$, i.e. to -0.268 and -3.732. The remaining four zeros and seven poles converge to 1 on the real axis regardless of the pole-zero location of the underlying continuous-time system.

In the case of $\delta-$ model, the sampling zeros converge to $-\infty$ as $T \to 0$, while the rest of zeros, as well as all poles converge to the continuous-time zeros and poles, respectively. The behavior of two sampling zeros in the delta model for $T = 0.1, 0.15, 0.20, \ldots, 1.0$ s is visualized in Fig. 4. It is shown that, under the fast sampling, one sampling zero (denoted by $\circ$) goes to $-1.27/T$ and the another zero (denoted by $*$) migrates to $-4.73/T$.

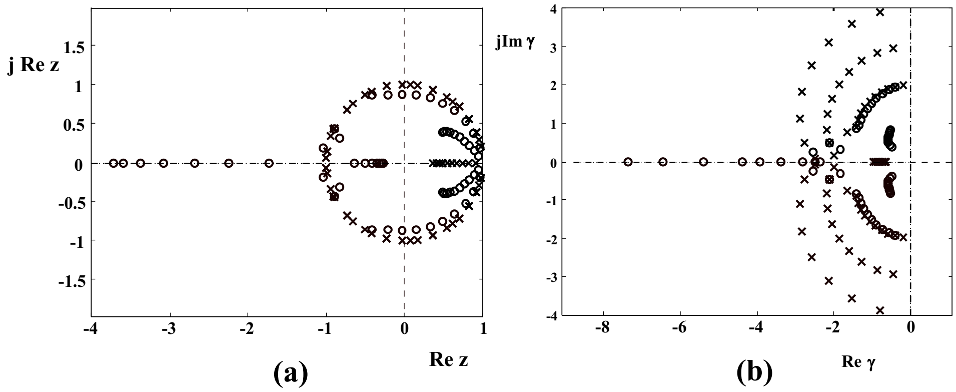


Fig. 3. Pole-zero variation of shift operator model and delta model with respect to sampling period.

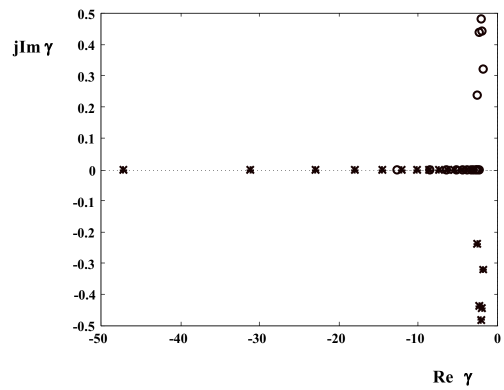


Fig. 4. Variation of delta model sampling zeros with respect to sampling period.

5 Pole Placement Design Based on $q-$ and $\delta-$ Object Model

A simple servo design method based on classical pole-zero assignment is chosen as a typical representative of analytical design techniques. The servo specifications are expressed in terms of a closed-loop system model that gives the desired response to command signals. However, it is of interest to compare the results of control system design based on $q-$ and $\delta-$ object model in a single-input single-output system given in Fig. 5, taking into account the quantization effects. Thus, the design problem may be interpreted as finding the polynomials R, S, and T with respect to the desired system continuous-time response. This leads to the application of the classical pole-placement algorithm with or without cancellation of process zeros [16]. In the considered control system the object transfer function is $1/s(s+1)$. The sampling periods of $0.25s$ and $1s$ are adopted. The speed of continuous-time closed-loop system response and stability margin are specified by

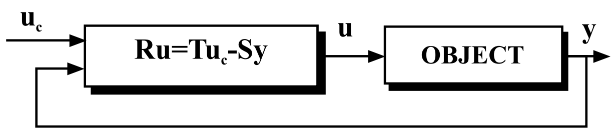


Fig. 5. Block diagram of a control system.

the relative damping coefficient $\zeta = 0.7$ and natural frequency $\omega_n = 1\text{rad}/\text{s}$ of the closed-loop system dominant pair of poles inside the principal strip of the $s-$ plane.

The control law can be written as

$$(q + r_1)\, u(t) = \left(t_0 q + t_1\right) u_c(t) - \left(s_0 q + s_1\right) y(t) \tag{22}$$

or

$$(\delta + \bar{r}_1)\, u(t) = \left(\bar{t}_0 \delta + \bar{t}_1\right) u_c(t) - \left(\bar{s}_0 \delta + \bar{s}_1\right) y(t)\ , \tag{23}$$

where $t \in \Omega(T)$. Two closed-loop system response specifications with and without cancellation of process zero for different sampling rates is considered. To carry out the design, it was necessary to solve some linear polynomial equations and the results are given in Table 2. The control system given in Fig. 5 has been

Table 2. Control parameters in relation (22) and (23)

	Pole placement	of closed-loop system		
	with zero cancellation		without zero cancellation	
	$T = 0.25s$	$T = 1s$	$T = 0.25s$	$T = 1s$
r_1	-0.9201	-0.7183	0.0538	0.2173
$\bar{r}_1$	7.6806	1.7181	0.4487	0.5199
t_0	1.8215	1.3485	0.9487	0.7848
$\bar{t}_0$	1.8229	1.3485	0.9497	0.7849
t_1	0	0	0	0
$\bar{t}_1$	7.2917	1.3485	0	0
s_0	4.3948	-2.5733	2.5284	1.0874
$\bar{s}_0$	4.3958	1.6782	0.5006	0.2649
s_1	1.6782	-0.3297	-1.5798	-0.3026
$\bar{s}_1$	7.2917	1.3485	0	0

simulated in all details, taking into account the quantization effects ($8-$ and $12-$bit conversion). The control law is implemented in accordance with (22) and (23) for different sampling periods. The simulation results are given in figures 6, 7 and 8.

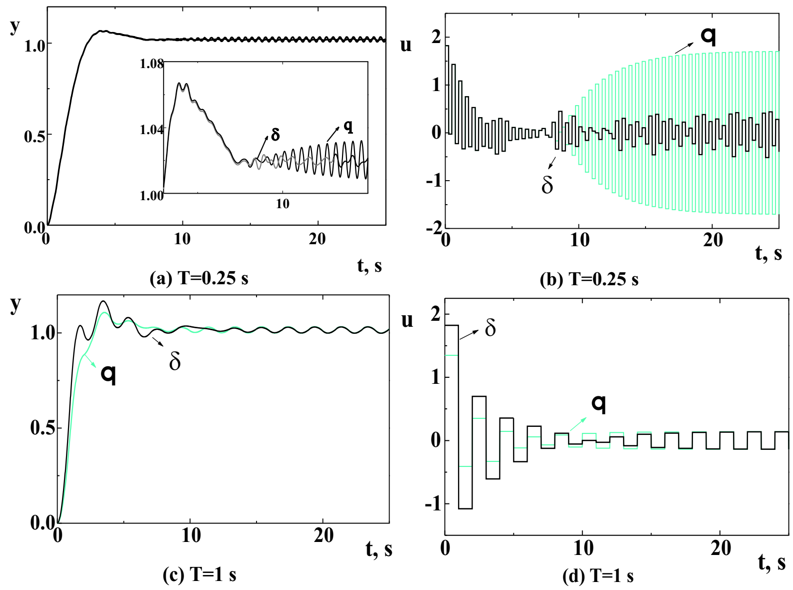


Fig. 6. Simulation results (8−bit conversion).

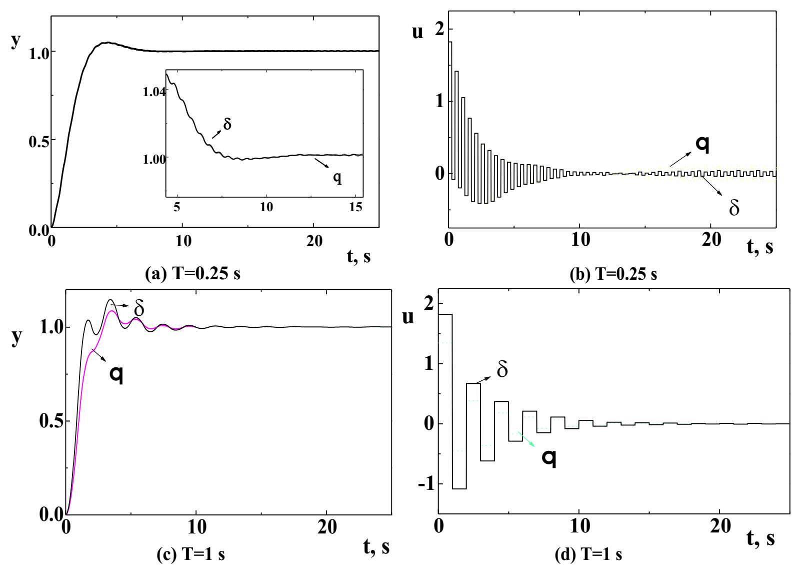


Fig. 7. Simulation results (12−bit conversion).

The appearance of the ”ringing” or the ”ripple” in the control signal is well-known. It is caused by the cancellation of the stable zero of discrete object $q-$model, located on the negative real axis [16]. The ripple is not noticeable in the output signal at the sampling instants (see figures 6(a)and 6(c)). However, it is seen as a ripple in the output signal between the sampling instants. The amplitude of the ripple in the output depends on the sampling period and goes down rapidly as the sampling period is decreased. Fig. 6 visualises that, by faster sampling, the ripple in the control and output signals are moderated by control based on the object $\delta-$model. The same regulation quality is preserved by 12-bit conversion, too. All, but somewhat little expressed, mentioned effects can be seen in Fig. 7.

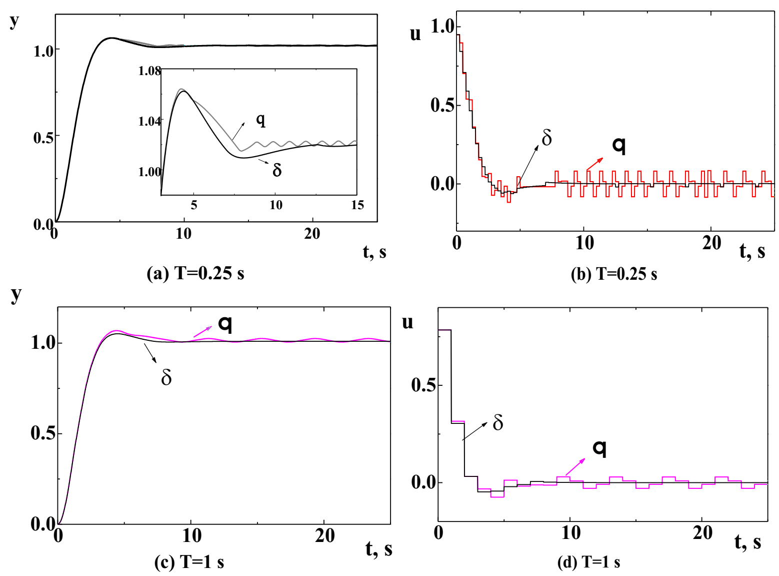


Fig. 8. Simulation results (12−bit conversion).

Fig. 8 illustrates the second procedure of pole placement based on $q-$ and $\delta-$model of the object with no cancellation of process zero. Notice that the control signal is much smoother and, at step change in the reference signal, twice less amplitude at $t = 0$. It can be seen that the delta domain design has clearly superior properties relative to the shift form.

6 Conclusion

In recent decades, an increasing attention has been given to the so-called $\delta-$operator owing to its wide applicability in many fields of engineering. This paper deals with the effects of the sampling time on the zero dynamics of continuous linear single input single output systems expressed in both $q-$ and $\delta-$operator forms. By $\delta-$domain modelling, the zeros introduced by discretization process (sampling zeros) are easily distinguished and, as the sampling time tends to zero, migrate to negative infinity. The conclusion, that under fast sampling in $\delta-$ model one can neglect the sampling zeros, is very important for delta domain control system design based on the classical pole-placement techniques with or without zero cancellation. To illustrate the control system performances that can be achieved in both $q-$ and $\delta-$ design domain, some simulation results have been given.

Acknowledgments

I feel deeply honored and I am very grateful for being invited to submit the paper for the special volume of this scientific Journal dedicated to Professor Milić R. Stojić on the occasion of his 65th birthday.

Professor Stojić has influenced me by both his teaching activities and his scientific work. I got to know Professor Stojić through his textbook of permanent value "Continuous-time automatic control systems", thirty years ago. Unfortunatly, as an undergratuate I did not attend his lectures, but afterwards I had many opportunities to assure myself of his teaching qualities par excellance. As his graduate student and assistant, I greatly benefited from courses and discussions with professor Stojić. He was the Supervisor of my M.S. and Ph.D. Thesis and I would like again, on this occasion, to express my gratitude to him for helping me in so many ways and arousing a passionate love for scientific work. He is truly a remarkable man, known to most of the world, and his books and papers were and remain the challenge to me.

I hope that professor Stojić will continue to share his life experience, the well-known sense of humor and inexhaustible energy and enthusiasm, with his collaborators and friends. On the occasion of his 65th birthday celebration in FACTA UNIVERSITATIS, Ser.: Electronics and Energetics, I would like to express my deep gratitude and sincere respect to Professor Stojić, with best wishes for his long life, good health and personal happiness.

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