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Some useful MATHEMATICA Teaching examples

Milan Tasić, Predrag Stanimirović, Ivan Stanimirović, Marko Petković, and Nebojsa Stojković

Abstract: We show how a computer algebra system in MATHEMATICA can be used in several elementary courses in mathematics for students. We have also developed an application in programming language DELPHI for testing students in MATHEMATICA.

Keywords: MATHEMATICA, teaching.

1 Introduction

MATHEMATICA is a powerful computer algebra system developed by Wolfram Research, and the main developer is Stephen Wolfram [1], [2]. It is a very high level programming language, adaptive to various types of courses in mathematics. MATHEMATICA is a system for doing mathematics by computer. It includes arbitrary precision and exact numerical computation, symbolic computation, graphics, sound, hyperlinked documentation and interprocess communication - all integrated together in one easy-to-use package. The MATHEMATICA computer algebra system is an attractive medium for teaching mathematics [3]. There exists several computer algebra systems, such as Maple,MatLab and others, and at this level it is mainly a matter of taste which system one chooses to use. However, looking a little further it is the authors opinion in [4] that MATHEMATICA is superior to the others for the following reasons: MATHEMATICA as a programming language is very structured and highly adaptive to a variety of applications, both technical and theoretical. MATHEMATICA

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M. Tasić is with University of Niš, Faculty of Technology, Bulevar oslobodjenja 124, 16000 Leskovac, Serbia and Montenegro (e-mail: milan12t@ptt.yu). S. Stanimirovis̀, M. Stanimirović and M. Petković are with University of Niš, Department of Mathematics, Faculty of Science, Višegradska 33, 18000 Niš, Serbia and Montenegro (e-mails pecko@pmf.ni.ac.yu, ywcha@yahoo.com, dexter_of_nis@neobee.net). N. Stojkoviś is with University of Niš, Faculty of Economics, Trg Kralja Aleksandra 11, 18000 Niš, Serbia and Montenegro (e-mail: nebojsas@orion.eknfak.ni.ac.yu).

has a well-defined protocol for interprocess communication. This protocol called MathLink is implemented as a C library. J/Link is built on Math-Link and allows the user to write Java code in MATHEMATICA and visa versa.

There is an initiative of the Section of Computer Science coordinated by Amilicar Sernadas to introduce the language MATHEMATICA as the first programming language (see the Web page http://www.cs.math.ist.utl.pt/cs/courses/mathematica.html). This decision dates back from 96/97 for: chemical engineering, materials engineering, environmental engineering, mining engineering. More recently, this approach was extended to: biological engineering, biomedical engineering, chemistry, computer science, industrial management and engineering. This choice was made taking into account the following advantages of MATHEMATICA over the classical alternatives (such as Pascal, C or FORTRAN90): interactive prototyping environment, data visualization tool, symbolic computation besides numerical computation, programming without concern for memory management, several paradigms of programming, such as functional, recursive, imperative, rule based. The students will also be exposed to a lower level programming language at a later stage of their curricula.

In teaching programming for students in mathematics courses, one of the important features for programming languages may be the ability to treat functions as higher order functions. This feature is presented in [5] and compared with programming language C.

MATHEMATICA can be embedded into webservers via an application mainly developed by Tom Wickham-Jones called webMathematica [6]. MATHEMATICA computing unit called the kernel is separated from the frontend and can be run on a powerful computer on a network where the workstations can run a frontend communicating with the kernel. MATHEMATICA support several different text formatting languages such as TEX, HTML, MathML and others.

In MATHEMATICA version 3.0 and later, we can use palettes. The most common request from students when applying MATHEMATICA in a classroom to avoid typing at all cost.

Due to above stated useful properties, MATEMATICA is applicable in almost all elementary courses in mathematics for students. In the second section we give several applications of the package MATHEMATICA in various courses for students. In the third section we are developed an application in the package Delphi to teach students in MATHEMATICA. In the last section we describe a MATHEMATICA package for teaching the graphical solution of two-dimensional linear problem.

2 A practise approach trough examples

We will first give a some examples how MATHEMATICA was used in various courses for students. In these short notes we, of course, can not cover all types of applications.

2.1 Gauss-Jordan elimination

In elementary linear algebra almost all problems boils down to solving some systems of linear equations. Hence it is important that the students are able to solve systems of this type.

In [4] the author illustrates how student can get training in the Gauss-Jordan method using some simple MATHEMATICA programs. The idea here is that the students only enter some coefficient matrix and then concentrate on elementary row operations. It is included a ”one step back” function in case of an error, but if the student do several errors in sequence, he has to start all over.

Example 2.1 The example is to solve a system of three equations with three unknowns with the following augmented matrix:

$$\begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & -3 & 1 & 3 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

The student can use three elementary line operations. The MATHEMATICA functions involved are given in [4].

LM[{{0, 1, -3, 1}, {1, -3, 1, 3}, {2, 2, 1, 0}}]

The matrix is:

$$\begin{pmatrix} 0 & 1 & -3 & 1 \\ 1 & -3 & 1 & 3 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

SL[1,2]

Switching line 1 and line 2:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

AL[1,3,-2]

Adding -2 times line 1 to line 3:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 8 & -1 & -6 \end{pmatrix}$$

AL[2,3,8]

Adding 8 times line 2 to line 3:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 16 & -25 & 2 \end{pmatrix}$$

Error.

RD[]

One step back to correct an error:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 8 & -1 & -6 \end{pmatrix}$$

AL[2,3,-8]

Adding -8 times line 2 to line 3:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 23 & -14 \end{pmatrix}$$

ML[3,1/23]

Multiplying line 3 with $\frac{1}{23}$:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & \frac{14}{23} \end{pmatrix}$$

AL[3,2,3]

Adding 3 times line 3 to line 2:

$$\begin{pmatrix} 1 & -3 & 1 & 3 \\ 0 & 1 & 0 & -\frac{19}{23} \\ 0 & 0 & 1 & -\frac{14}{23} \end{pmatrix}$$

AL[3,1,-1]

Adding -1 times line 3 to line 1:

$$\begin{pmatrix} 1 & -3 & 0 & \frac{83}{23} \\ 0 & 1 & 0 & -\frac{19}{23} \\ 0 & 0 & 1 & -\frac{14}{23} \end{pmatrix}$$

AL[2,1,3]

Adding 3 times line 2 to line 1:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{26}{23} \\ 0 & 1 & 0 & -\frac{19}{23} \\ 0 & 0 & 1 & -\frac{14}{23} \end{pmatrix}$$

The system is now solved, and the student can check his result using **Solve:**

Solve[{y-3z==1,x-3y+z==3,2x+2y+z==0},{x,y,z}]

$\{\{x \to \frac{26}{23}, y \to -\frac{19}{23}, z \to -\frac{14}{23}\}\}$

The MATHEMATICA functions referred to in this note are given in [4].

2.2 Linear Equations in Two Unknowns

Look at a single linear equation in two unknowns:

$$x + by = c$$

This is the equation of a line in the plane, provided the coefficients *a* and *b* are not both zero. In the next example two such equations are solved simultaneously and MATHEMATICA is used to look at their common solutions [7].

Example 2.2 Carry out Gaussian elimination by hand to solve the following system of linear equations:

$$\begin{aligned} x + 4y &= 6 \\ 3x - y &= 5 \end{aligned}$$

Then use MATHEMATICA to draw the two lines, and find the location of the solution on the plot.

Solution: Carry out Gaussian elimination by hand.

The augmented matrix for this system is $\begin{pmatrix} 1 & 4 & 6 \\ 3 & -1 & 5 \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix}$.

An application of back substitution yields $(x,y) = (2,1)$ as the unique solution. To draw the lines corresponding to the above linear equations, we first enter two corresponding symbolic equations and label them by **eqn1** and **eqn2**, respectively:

$$\begin{aligned} eqn1 &= x + 4y == 6 \\ eqn2 &= 3x - y == 5 \end{aligned} \tag{2.1}$$

Then we draw the lines contained in (2.1) by using the standard command DrawLines:

DrawLines[{eqn1, eqn2}, {x, y}]

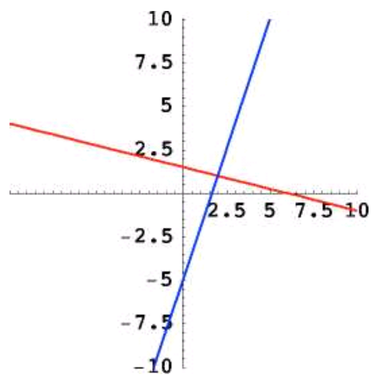

Fig. 1. Graphics

Note that these two lines intersect at the point $\{2,1\}$, as we expected.

Exercise 1: (a) Replace the second equation above (the one called **eqn2**) by a new equation so that the system {**eqn1, eqn2**} has no solutions. Begin by describing how the two lines should be related to one another geometrically.

(b) Check you answer by using **DrawLines** and by carrying out Gaussian elimination by hand.

Answer to Exercise 1

We can find a new **eqn2** by using the same coefficients of x and y as in **eqnl** (and hence creating a line with the same slope) but changing the right side:

$$eqn2 \quad = x + 4y == 0 \qquad (* \text{ corresponding to } x + 4y = 0 *)$$

A MATHEMATICA plot confirms that these lines are parallel:

DrawLines[{eqn1,eqn2},{x,y}]

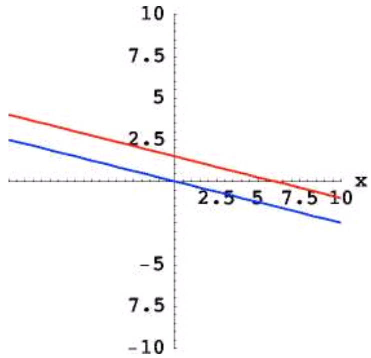

Fig. 2. Graphics

The augmented matrix for this system is $\begin{pmatrix} 1 & 4 & 6 \\ 1 & 4 & 0 \end{pmatrix}$, which reduces to $\begin{pmatrix} 1 & 4 & 6 \\ 0 & 0 & -6 \end{pmatrix}$. The second row shows that we have an inconsistent system.

Now let's consider three equations in two unknowns. First, let's go back to the original two equations; then we will consider what can happen when we include a third equation.

$$\begin{aligned} eqn1 &= x + 4y == 6 \\ eqn2 &= 3x - y == 5 \end{aligned}$$

Exercise 2: (a) Create a third equation (call it eqn3) so the system {eqn1, eqn2, eqn3} has a unique solution, but no two of the three lines are equal. Begin by describing how the three lines should be related to one another geometrically.

(b) Check your answer by using DrawLines.

Answer to Exercise 2

Choose the third line so it goes through the point of intersection $(2,1)$ of the first two lines. For example:

$$eqn3 = \quad x - 2y == 0 \qquad (* \text{ corresponding to } \quad x - 2y = 0 *)$$

We check with a MATHEMATICA plot that the three lines intersect at this single point.

DrawLines[{eqnl,eqn2,eqn3},{x,y}]

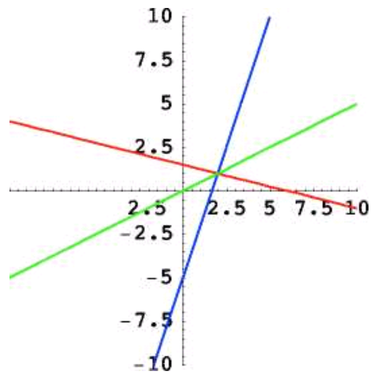

Fig. 3. Graphics

2.3 Tangent line problem

A typical problem given to students is ”Find an equation of the line tangent to the graph of $f(x) = x^2$ at the point $(2, f(2))$”. With the advent of computer algebra systems and their excellent graphics capabilities, a new type of tangent line problem is accessible to students taking calculus. We will begin by stating the problem in general, and then work out a specific example. The problem can be stated as follows: ”Let $f(x)$ be a differentiate function. Under what conditions does there exist a differentiable function $g(x)$ such that the lines tangent to the graph of y = g(x) are of the form $y = ax + f(a)$ for all a in the domain of f [8]? If $g(x)$ exists, then find an expression for it in terms of $f(x)$.

Example 2.3 Let $f(x) = x^2$. Is there a differentiable function g(x) whose tangent lines are of the form $y = ax + a^2$ for all $a \in \mathbb{R}$? If so, what is $g(x)$

Solution: The graphs of the functions $y = ax + a^2$ can be generated by the programs listed below.

```
Clear [t];t = Table[a*x + a^2,{a, -10,10,0.1}];
Plot[Evaluate[t],{x,-10,10}, PlotRange->{-10,10},AspectRatio->1]
```

The picture shown on the computer suggests that the function $g(x)$ exists and $g(x)$ is a quadratic function of the form $g(x) = -\alpha x^2$, where $\alpha > 0$. The equation of the line tangent to the graph of $y = g(x)$ at the point $(\beta, g(\beta))$ is given by $y = -2\alpha\beta + \alpha\beta^2$. This tangent line is of the form $y = ax + a^2$ if only if $a = -2\alpha\beta$ and $a^2 = \alpha\beta^2$. Since $\alpha > 0$ we see that these equations are satisfied when $\alpha = \frac{1}{4}$. Hence the function $g(x) = -\frac{1}{4}x^2$ has tangent lines of the form $y = ax + a^2$. We would like to point out that without the ability to visualize the set of lines of the form

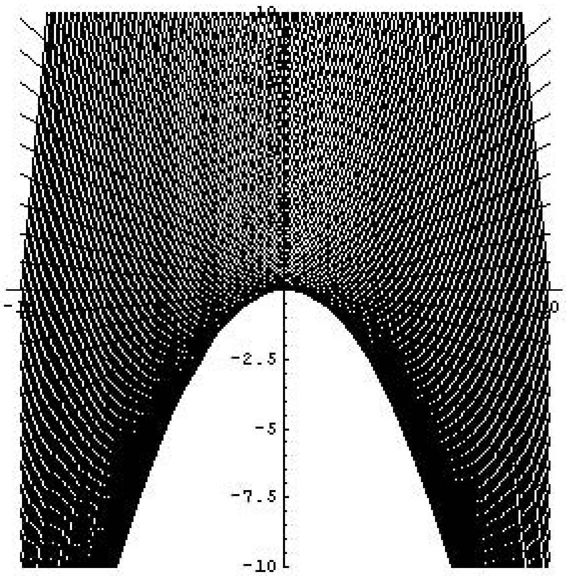

Fig. 4. The existence of function $g(x)$

$y = ax + a^2$, this problem would be extremely difficult for students to solve. Hence the computer aids in the solution of this problem by allowing us to visualize the problem and then make a conjecture about the existence and form of the function $g(x)$ that we are trying to find.

2.4 Venn Diagrams

In the paper [9] it is described an innovative contribution to flexible learning, using MATHEMATICA in an interactive package. Unlike previous approaches based on MATHEMATICA, there are no need for students to learn MATHEMATICA, a process which may obscure the mathematics.

Example 2.4 Venn diagrams: One of intentions from [9] is to help students make links between different representations of the same concept-in this case between visual and symbolic representations. When the student clicks on the button, MATHEMATICA randomly generates a Venn diagram such as the one shown in Figure 2. The student is asked to express the colored region in set theoretic notation.

The student's answer is entered via the keyboard together with the use of a symbol palette. MATHEMATICA then represents the answer graphically as a second Venn diagram. If the student has typed any one of the many logically correct expressions-such as $(C \setminus B) \cup ((A \cap B) \setminus C)$ or $((A \cap B) \cup C)\ (B \cap C)$ - then the two diagrams will

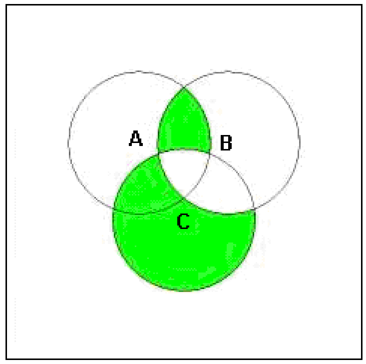

Fig. 5. Venn Diagram

be identical, so that the feedback is immediate. Even if the student enters an incorrect symbolic expression, he will have an immediate graphical representation of this wrong answer, which helps to provide understanding of why the answer is incorrect. Here even an incorrect answer has provided a positive learning experience. The two diagrams will match only when a correct answer is given.

2.5 WebMathematica

WebMathematica is a new web-based technology developed by Wolfram Research that allows the generation of dynamic web content with MATHEMATICA. It combines the computational engine of MATHEMATICA (the MATHEMATICA kernel) with web pages that are written in the HTML language and creates a synergism that is a useful tool for enhancing teaching mathematics and mathematically oriented topics. With this technology, the distance students should be able to explore and experiment with some of the mathematical concepts.

One of the most exciting new technologies for dynamic mathematics on the World Wide Web is a webMathematica. This new technology developed by Wolfram research enables instructors to create web sites that allow users to compute and visualize results directly from the web browsers. This is achieved by integrating the MATHEMATICA computer algebra system with the latest web server technology. The students use the existing Internet browsers such as Internet Explorer or Netscape as an interface to webMathematica and they do not need to know MATHEMATICA and install the program in their machine to use it. WebMathematica is based on a core technology called MATHEMATICA Server Pages (MSP) [6]. MSP technology allows a site to contain HTML pages, which are enhanced by the addition of MATHEMATICA commands. When a request is made for one of these pages, which are called

MSP scripts, any MATHEMATICA commands are evaluated and the computed result is placed in the page. Through webMathematica the instructors and students can fully utilized the computational power of MATHEMATICA for pedagogical applications.

3 One application in Delphi to test students

One of the methods used to reduce the students key typing is to distribute a prepared Notebook. The teacher can broadcast instructions to the students, which in turn the students will read and respond to, and then both teacher and students can join together in a live discussion on a topic of choice from within their MATHEMATICA session.

We have developed a package for learning MATHEMATICA, but the ideas are applicable to other areas. Our application are also used for teaching Microsoft Office programs such as Word and Excel, and for practise exam of these programs. The student requires only basic computer skills and the ability to use standard tools such as hyper links and buttons. All questions and results are placed on the bottom of screen and students workspace remain on top. In this way it is possible to concentrate on the concepts and provide a powerful learning tool.

The notebooks explain the concepts and definitions and provide examples and exercises for the student.

Teacher can wrote in html file (or mht file) instructions to the students at the bottom of screen Students can type answer at the top in Notebook file and respond to teacher. The bottom of the page can be organized as check box area with several answers, where only one answer is true.gh wrong answer.

Code sample on button for next question.

```
procedure TTEST.Button1Click(Sender: TObject); begin
 if NumberOfQuestion<5 then NumberOfQuestion:=NumberOfQuestion+1;
 URLs:=PChar(GetCurrentDir+'\Question\m'
                          + inttostr(NumberOfQuestion)+'.mht');
 findaddress;
 Label1.Caption:='Problem '+inttostr(NumberOfQuestion);
 try
 ShellExecute(Handle,'open',PChar('c:\MathTest\Question\Problem.nb'),
                                      nil,nil,SW_SHOW);
  Finally
 end;
end;
```

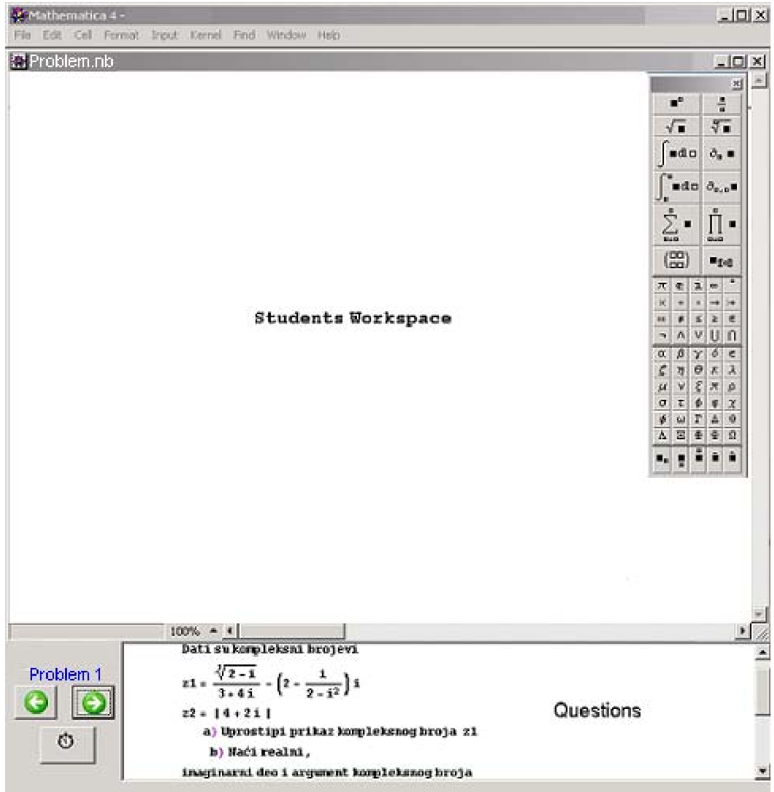

Fig. 6. Test application screen

4 Graphical solution of two-dimensional linear problem

Consider the standard form of two-dimensional linear program:

$$\begin{aligned} \text{Maximize} \quad & c_1 x + c_2 y + d, \\ \text{Subject to} \quad & a_{i1} x + a_{i2} y \leq b_i, \ i = 1, \ldots, m \\ & x \geq 0, y \geq 0 \end{aligned} \tag{4.1}$$

Since the objective function is of two variables, it can be applied well known graphical procedure for solving the linear programming problems [10]. If the restricting conditions in (4.1) are given in the form of inequalities, each of the corresponding straight lines divides the area into a range which is possible for these conditions and a range impossible for these conditions. The permissible conditions are located in the range P that is permissible for all conditions (the region of feasible solution). The optimal solution is found by drawing the graph of the modified objective function $f(x,y) = 0$ and parallel shifting of this in the direction of the gradient vector (c_1, c_2). The optimal solution is unique if the straight line $f(x,y) = f_{max}$ runs trough a corner point of the possible range. In minimization, the straight line must be shifted in the opposite direction. The following code implements the graphical procedure. Since it animates the parallel shifting of the line $f(x,y) = 0$, it is useful for teaching purposes.

```
Geom[f_, g_List] := Module[{res = {}, res2 = {}},
   var = Variables[f];
   p2 =InequalityPlot[g, {var[[1]]}, {var[[2]]}, AspectRatio -> 1,
        DisplayFunction -> Identity];
   h = g /. {List -> And};
   h = InequalitySolve[h, var];
   If [h == False, Print["Problem is infeasible!!!!"]; Break[]; ];
   g1 =g /. {LessEqual -> Equal, GreaterEqual -> Equal,
        Less -> Equal,Greater -> Equal};
   For[i = 1, i <= Length[g1] - 1, i++,
       For[j = i + 1, j <= Length[g1], j++,
           t = FindInstance[g1[[i]] && g1[[j]] && h, var];
           If[t != {}, AppendTo[res, {t[[1, 1, 2]], t[[1, 2, 2]]}];
                  AppendTo[res2, {ReplaceAll[f, t[[1]]], t[[1]]}];
           ];
       ];
   ];

   For[i = 1, i <= Length[res2] - 1, i++,
     For[j = i + 1, j <= Length[res2], j++,
       If[res2[[i, 1]] > res2[[j, 1]],
         r = res[[i]]; res[[i]] = res[[j]]; res[[j]] = r;
         r = res2[[i]]; res2[[i]] = res2[[j]]; res2[[j]] = r;
       ];
     ];
   ];

   p1 =ListPlot[res, PlotStyle -> {PointSize[0.025], Hue[1]},
       DisplayFunction -> Identity];
   r = Solve[f == 0, var[[2]]][[1, 1, 2]];

   mx = Max[Table[res[[i, 1]], {i, 1, Length[res]}]];
   my = Max[Table[res[[i, 2]], {i, 1, Length[res]}]];

   For[i = 1, i <= Length[res], i++,
       n = ReplaceAll[y - r, res2[[i, 2]]]
   ];

   cf[i] = Plot[r + n, {x, -mx/10, mx*1.1},
       PlotStyle -> {Thickness[0.007], Hue[1]},
       DisplayFunction -> Identity];
```

```
    ];

    ShowAnimation[
    Table[Show[p2, p1, cf[i], AspectRatio -> 1,
         PlotRange -> {{-mx/10, mx*1.1}, {-my/10, my*1.1}}],
             {i, 1, Length[res], 1}]
    ];
    If[res2[[Length[res2], 1]] == res2[[Length[res2] - 1, 1]],
    Print["Optimal solution is given by \[Lambda]*",
         res[[Length[res]]],
         "+(1-\[Lambda])*", res[[Length[res] - 1]],
         ",    0<=\[Lambda]<=1"],
             Print["Optimal solution is: ", res[[Length[res]]]];
    ];
];
```

First we plot the constraints set (graphics `p2`) applying the standard MATHEMATICA function `InequalityPlot`. Then in next for loop we are finding all extreme points lying in the feasible region, using the standard function `FindInstance`, new in version 5.0. These points are plotted on the graphics `p1`. In the next step we sort these points by the value of the objective function, and shift in parallel the line $f(x,y) = 0$ in the direction of its gradient vector (c_1, c_2) through the sorted extreme points (the array of graphics `cf[i]`). Finally using the function `ShowAnimation` we animate plotted graphics.

Example 4.1 For example

```
Geom[8x+12y, 8x+4y <= 600, 2x+3y <= 300, 4x+3y <= 360, 5x+10y >= 600, x >= 0, y >= 0]
```

we generate the following graphics included in the animation:

```
Optimal solution is given by λ*{30,80}+(1-λ)*{0,100}, 0≤ λ ≤1
```

5 Conclusions

The major difficulty in teaching mathematics comes from necessity of striking a balance between the connections to real life situations and the great volume of technical knowledge required to succeed in the science itself. In recent years, the use of technology in education has become an important issue. It is too much to expect that all students will be able to understand the most complicated material through reading and lectures only. The best classrooms, are those where both the teacher and the students learn from each other.

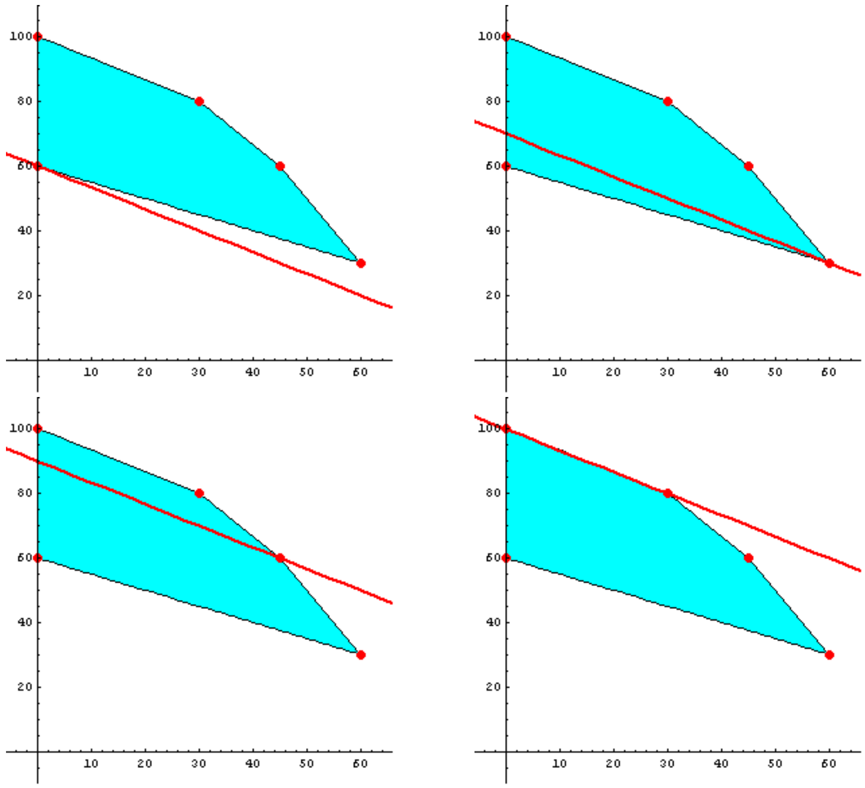

Fig. 7.

We describe several teaching materials which assist students to make connections between different representations of the same concept - verbal, graphical and algebraic.

It is well known that students learn more quickly, and with less pain, when concepts can be demonstrated interactively. Problems that require visual representation like graph, diagrams, animations and moving images can be solved with webMathematica that respond to students questions, answers or commands.

The learning experiences must be well organized and integrated in a comprehensive modular approach to facilitate for continuous and student-centered-learning. The design of instruction is by far the most important parameter in an effective teaching and learning.

Before the advent of MATHEMATICA, students who were not proficient in calculations spent a majority of their time performing the computations, and very little time analyzing and processing the results. Due to MATHEMATICA application, students take much more time for analyzing and processing the results.

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