

Three Theorems on Hierarchical Decomposition of Similarity Linear Systems

This paper is dedicated to Prof. K. Tröndle on the occasion of his retirement

**Yuan-Wei Jing, Jun Zhao, Mile J. Stankovski,
and Georgi M. Dimirovski**

Abstract: In this paper the problem of system decomposition of complex linear dynamical systems by exploiting the similarity property is studied. System decompositions are sought in terms of similarity hierarchical structures. The method for constructing the transformation is derived. The conditions for such decomposition of complex linear systems are given.

Keywords: Circuits and systems theory, controllability, isomorphic decomposition, similarity linear systems.

1 Introduction

In the contemporary systems engineering [1], most of modern control systems are rather complex in their engineering design, and having some distinctive properties such as hierarchy, similarity, symmetry, etc. [2]-[10]. It is well known that in the study of the steady-states and the dynamics of the complex systems, the analysis of the structure is rather important [11, 12]. This problem is the essential one even in the class of linear complex dynamical systems [13]-[17]. Application examples

Manuscript received January 10, 2004.

J.-W. Jing and J. Zhao are with Northeastern University, Institute of Control Science, School of Information Science & Engineering, Shenyang, Liaoning, 110006, P.R. of China (e-mail: ywjing@etang.com and zdongbo@pub.cninfo.com). M. Stankovski and G. Dimirovski are with SS Cyril & Methodius University, Institute of Automation & Systems Engineering, Faculty of Electrical Engineering, MK-1000 Skopje, Republic of Macedonia (e-mail: milestk@etf.ukim.edu.mk). G. Dimirovski is with Dogus University, Department of Computer Engineering, Faculty of Engineering, Acibadem - Kadikoy, 81010 Istanbul, Republic of Turkey (e-mail: gdimirovski@dogus.edu.tr).

that may well be modelled usefully by means of both complex or composite linear systems are the control system structures for socio-economic, mechatronic and robotic systems, just to mention a few [1, 14, 10].

In this paper, a class of hierarchical, similarity structure systems is dealt with following the previous results in [16, 7, 10]. The problem of feasible decompositions of similarity structure composite linear systems is addressed. In doing so, the conditions for transforming a system into the one of hierarchical similarity structure are discussed in more detail, and some novel results derived. Firstly, the concept of hierarchical structure similar systems is presented with regard to practical applications in control problems. The existence of coordinate transformation by means of which the system can be decomposed into structure similar systems is explained by using the concepts of eigenspace and eigenvectors. The method for constructing the transformation needed is derived via deriving the proofs of the new theorems presented.

2 On Hierarchical Similarity Structure Systems

Consider the following controlled composite system $S \supset S_i, i = 1, \dots, n$ described by equations

$$\begin{aligned} \dot{x}_1 &= A_1 x_2 + B_{11} u_1 \\ \dot{x}_2 &= A_2 x_3 + B_{21} u_1 + B_{22} u_2, \\ &\vdots \\ \dot{x}_{k-1} &= A_{k-1} x_k + B_{k-1, k-2} u_{k-2} + B_{k-1, k-1} u_{k-1} \\ \dot{x}_k &= A_k x_k + B_{k, k-1} u_{k-1} + B_{k, k} u_k \end{aligned} \quad (1)$$

In here, $A_i, i = 1, \dots, k$, are the square matrices with dimension $n_k, kn_k = n$, and r , the dimension of input u , is not least than k . Such a composite system possesses the hierarchical similar structure in which the state x_i is only related to the state x_{i+1} and the control variables u_{i-1} and $u_i, i = 2, \dots, k-1$.

Hence for linear dynamical systems with similarity property, Eqs. (1), the following definitions [16, 7, 10] are well posed and particularly useful. These are used in the sequel.

Definition 2.1 : For two subsystems S_i, S_j in a system S , if they have the forms as

$$\dot{x}_i = A_i x_{i+1} + B_{i, i-1} u_{i-1} + B_{ii} u_i \quad (2)$$

$$\dot{x}_j = A_j x_{j+1} + B_{j, j-1} u_{j-1} + B_{jj} u_j \quad (3)$$

then, the two subsystems S_i, S_j are said to be structure similar. In particular, they are completely structure similar when $A_i = A_j$

In comparison with the others, the last subsystem S_k , in the system (1), does not have direct relation to other subsystems, and the first subsystem S_1 has only input variable u_1 . Hence, in manipulation robotics, for instance, we can take the subsystem S_1 as a central controller of the robot systems, and the subsystem S_k as an operational hand or terminal.

Definition 2.2 : *If all of the subsystems S_i in the system $S, i = 1, \dots, k = 1$, are structure similar, then the system S is said to be a hierarchical similarity structure or hierarchical structure similar system.*

There are many advantages in using this kind of systems. In the sequel, we recall some important previous results that are found elsewhere [3, 16] in the literature.

Theorem 2.1 : *Assume that system S is hierarchical structure similar. Then the system S possesses stability property if, and only if, the subsystem S_k*

$$\dot{x}_k = A_k x_k + B_{k,k-1} u_{k-1} + B_{k,k} u_k$$

is stable.

Theorem 2.2 : *Assume that system S is hierarchical completely structure similar. Then the system S possesses controllability property if, and only if, the subsystem S_k is controllable.*

In practical applications, there are a number of systems with the property of being hierarchical similar as mentioned in the introduction. However, because of the selection of coordinate of the states, they often appear to be very commonly formulated, which are not alike to the system (1). A naturally arising question is what kind of systems can be transformed into the form described by Eqs. (1) and how to transform them. In other words, what are the conditions under which the complex linear systems can be decomposed into hierarchical systems with similarity structure.

3 Hierarchical Similarity Structure System Decomposition

For the class of control systems

$$\dot{x} = Ax + Bu \tag{4}$$

where $A, B \in R^{n \times n}$, $x \in R^n$, $u \in R^n$, the goal of present investigation is to find a non-singular matrix T so that, by using the transformation $x = Tz$, the system (4)

can be decomposed into a system possessing particular structure described as given below. Namely, the equivalent description sought is the following one

$$\dot{z} = \bar{A}z + \bar{B}u \quad (5)$$

where

$$\bar{A} = \begin{bmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} \bar{b}_{11} & 0 & 0 & \cdots & 0 & 0 \\ \bar{b}_{21} & \bar{b}_{22} & 0 & \cdots & 0 & 0 \\ 0 & \bar{b}_{32} & \bar{b}_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \bar{b}_{n-1,n-1} & 0 \\ 0 & 0 & 0 & \cdots & \bar{b}_{n,n-1} & \bar{b}_{nn} \end{bmatrix}$$

It is well known that a linear dynamical system can be decomposed in different forms by means of appropriate coordinate transformations. The system structure decomposition can lead to an easier case when dealing with systems in order to arrive at a structure with desired properties. A number of results have been reported on the systems decompositions and on the control of decomposed systems in particular; for instance, see [15, 4, 16, 17, 7, 8] and references therein. Nonetheless, in the literature only fewer studies are found on the similarity structure system decompositions the most comprehensive one being [10]. This paper is devoted solely to the study the similarity structure decomposition of composite linear systems following [10]. In what follows a couple of lemmas on previous results in linear system theory, compiled from the common literature, shall be needed.

Lemma 3.1 : *If there exists a non-singular transformation T so that the system (4) can be decomposed into (5), then A has the same eigenvalue as \bar{A} , and they are 0 and a_n .*

Lemma 3.2 : *If the system (4) can be decomposed into the equivalent systems (5), then*

1. *the matrix is singular;*
2. *the matrix cannot be transformed into a diagonal matrix.*

Remark 3.1: *If a system can be transformed into a system having a diagonal state matrix, as a matter of fact, the transformed system is state uncoupled actually. The Lemma 3.2 reveals the difference between the state diagonal systems and the hierarchical similar systems. These two types of linear composite systems have different properties with respect to their control and system design.*

Let $B = [B_1, B_2, \dots, B_n]$, $T = [T_1, T_2, \dots, T_n]$, where $B_i \in R^n$, $T_i \in R^n$, $i = 1, 2, \dots, n$.

Lemma 3.3 : Assume that $\text{rank}(A) = p$. If there exists a non-singular T , so that $\bar{A} = T^{-1}AT$, then T_1, \dots, T_{n-p} are the eigenvectors of A on $\lambda = 0$, while T_i is the eigenvector of $A^{j-(n-p-1)}$ on $\lambda = 0$, which satisfy

$$\begin{aligned} AT_j &= a_{j-1}T_{j-1}, \quad j = n-p+1, \dots, n-1, \\ AT_n &= a_{n-1} + a_nT_n. \end{aligned}$$

A necessary and sufficient condition for the existence of non-singular matrix T is given in terms of the subsequent novel theorems, the main theoretic results of this paper.

Theorem 3.1 : For the system (4), assume that $\text{rank}(A) = p$, a non-singular matrix T_n can be found to transform the state matrix A of system (4) into hierarchical similar structure if, and only if, the dimension of the eigen-subspace of A on $\lambda = 0$ is $n-p$, and there exists a set of vectors T_j, T_n satisfying

$$\begin{aligned} AT_j &= a_{j-1}, \quad j = n-p+1, \dots, n-1, \\ AT_n &= a_{n-1} + T_{n-1} + a_nT_n, \end{aligned}$$

respectively.

Theorem 3.2 : Assume that $\text{rank}(A) = p$ and the conditions as in Theorem 3.1 hold. If the column vectors B_1, \dots, B_{n-1} of B can be linearly determined by two of T_1, \dots, T_n , and B_n by T_n , then B can be transformed into \bar{B} with T .

The above presented study and its main results have led us to the following existence theorem.

Theorem 3.3 : For the system (4), if the state transition matrix, A , and control input matrix, B , satisfy the conditions in Theorem 3.1 and Theorem 3.2, respectively, then there exist a non-singular matrix T_n so that the system (4) can be decomposed into the form (5) by means of the transformation $h = Tz$.

The proofs of these theorems are given in the Appendix.

The theorems presented above give the conditions for the existence of non-singular transformation T while the respective proofs present the method for constructing matrix T . A system that has been hierarchically decomposed with the similarity structure can be studied more easily with respect to some of its properties, for instance, the stability, see [15, 4, 17].

4 Conclusion

This work has been devoted to a thorough investigation of the problems of hierarchical similarity structure systems decomposition. On the grounds of Zhang's theoretical study [10], three new theorems have been proved. By means of employing the concept of eigenvectors, the existence conditions for non-singular transformation matrix T have been derived. In addition, a method of constructing this matrix T while proving the new theorems has been derived too.

In applications, either in theoretical studies or in systems and control engineering practice, it appears always rather significant that a composite large-scale system be decomposed into hierarchical system with a similarity structure. For it enables alternatives to systems design tasks. These are topics for future research.

Acknowledgement

Authors express their thanks to Academician Si-Ying Zhang for his valuable advices in the study of complex systems with similarity and symmetry properties.

Appendix

Proof of Lemma 3.1

According to the condition $\bar{A} = T^{-1}AT$, we have $AT = T\bar{A}$. That is

$$\begin{aligned} AT &= [AT_1, AT_2, \dots, AT_n] \\ &= [T_1, T_2, \dots, T_n]\bar{A} = T\bar{A} \end{aligned}$$

Therefore it follows:

$$\begin{aligned} AT_1 &= 0, \\ AT_2 &= a_1 T_1, \\ &\dots, \\ AT_{n-p} &= a_{n-p-1} T_{n-p-1}, \\ AT_{n-p+1} &= a_{n-p} T_{n-p}, \\ &\dots \\ AT_{n-1} &= a_{n-2} T_{n-2} \\ AT_n &= a_{n-1} T_{n-1} + a_n T_n. \end{aligned}$$

Because of the similar structure of A and \bar{A} , $\text{rank}(A) = \text{rank}(\bar{A}) = p$. We can determine the rank of \bar{A} by the using the particular structure of it. Without loss of generality, let $a_1 = \dots = a_{n-p-1} = 0$. Then, a_{n-p}, \dots, a_{n-1} are some non-zero

constants. Hence $AT_1 = 0, \dots, AT_{n-p} = 0$. This is to say that T_1, T_2, \dots, T_{n-p} are the eigenvectors of A for the eigenvalue $\lambda = 0$. Also, we can infer that $A^2 T_{n-p+1} = a_{n-p} AT_{n-p} = 0$ due to $AT_{n-p+1} = a_{n-p} T_{n-p}$. Therefore, T_{n-p+1} is the eigenvector of A_2 on $\lambda = 0$. By analogy, the rest of vectors $T_{n-p+2}, \dots, T_{n-1}$ are, respectively, the eigenvectors of A^3, \dots, A^p for the eigenvalue $\lambda = 0$.

Proof of Theorem 3.1

Assume that the eigen-subspace of A on $\lambda = 0$ has dimension $n - p$. In this eigen-subspace there exists a set of coordinates T_i satisfying $AT_i = 0, i = 1, \dots, n - p$. It is obvious that T_1, \dots, T_{n-p} are linear independent. From the assumption, it follows that $AT_{n-p+1} = a_{n-p} T_{n-p}$. We now show that T_{n-p+1} is linear independent of T_1, \dots, T_{n-p} . If there are $h_1, h_2 \in R$ such that

$$h_1 T_{n-p} + h_2 T_{n-p+1} = 0, \quad (6)$$

then

$$A(h_1 T_{n-p} + h_2 T_{n-p+1}), \quad (7)$$

or

$$h_1 AT_{n-p} + h_2 AT_{n-p+1} = 0. \quad (8)$$

Because T_{n-p} is a nonzero eigenvector of A on $\lambda = 0$ with the result $AT_{n-p} = 0$, we can see that $h_2 AT_{n-p+1} = 0$ from (4). Therefore, $h_2 a_{n-p} T_{n-p} = 0$. Thus, $h_2 = 0$. Substituting it into (3) leads to $h_1 = 0$. It is to say that T_{n-p+1} is linear independent of T_{n-p} . It can be shown, by the same procedure, that T_{n-p+1} is linear independent of T_1, \dots, T_{n-p-1} . To T_{n-p+1} and T_{n-p+1} , if

$$r_1 T_{n-p+1} + r_2 T_{n-p+2} = 0 \quad (9)$$

where $r_1, r_2 \in R^1$, and then $A^2(r_1 T_{n-p+1} + r_2 T_{n-p+2}) = 0$. That is

$$\begin{aligned} r_1 A^2 T_{n-p+1} + r_2 A^2 T_{n-p+2} &= 0 \\ r_1 A(a_{n-p} T_{n-p}) + r_2 A(a_{n-p+1} T_{n-p+1}) &= 0 \\ r_1 a_{n-p} AT_{n-p} + r_2 a_{n-p+1} AT_{n-p+1} &= 0 \end{aligned} \quad (10)$$

This leads to $r_2 a_{n-p+1} AT_{n-p+1} = 0$ due to AT_{n-p} . Therefore we can get $r_2 = 0$. Substituting it into (5) leads to $r_1 = 0$. In other words, T_{n-p+2} and T_{n-p+1} are linear independent. And, then we can show that T_{n-p+2} is linear independent of T_1, \dots, T_{n-p} . By analogy, it can be shown that T_1, \dots, T_{n-1} are $n - 1$ vectors having the property of linear independence. Provided T_n that satisfies $AT_n = a_{n-1} T_{n-1} +$

$a_n T_n$ is not an eigenvector of A, A^2, \dots, A^p on $\lambda = 0$, it must be linear independent of T_1, \dots, T_{n-1} . Let $T = [T_1, \dots, T_n]$. Then T is non-singular and leads to $\bar{A} = T^{-1}AT$. To this end, the sufficiency has been proved.

The proof of the necessity part of conditions for Theorem 3.1 is obtained from Lemma 3.1 directly.

Proof of Theorem 3.2

Without loss of generality, let us assume

$$\begin{aligned} B_1 &= \bar{b}_{11}T_1 + \bar{b}_{21}T_2 \\ B_2 &= \bar{b}_{22}T_2 + \bar{b}_{32}T_3 \\ &\vdots \\ B_{n-1} &= \bar{b}_{n-1,n-1}T_{n-1} + \bar{b}_{n,n-1}T_n \\ B_n &= \bar{b}_{nn}T_n \end{aligned} \tag{11}$$

It follows at once that $B = TB$, which ends up the proof.

Proof of Theorem 3.3

The proof of this theorem follows at once from Theorems 3.1 and 3.2 and the Lemmas presented beforehand.

References

- [1] G. M. Dimirovski, "Control and supervision of complex processes: A challenge to systems engineering (Invited Plenary Lecture)," Invited Plenary Lecture Proc. TOK'98 of the 40th Anniversary of Turkish IFAC NMO. Istanbul: Turkish Automatic Control Committee and Istanbul Technical University, Oct. 1998, pp. 3–32.
- [2] J. W. Grizzle and S. I. Marcus, "Optional control of systems possessing symmetries," *IEEE Trans. Automatic Control*, vol. 29, no. 11, pp. 1037–1040, 1984.
- [3] —, "The structure of nonlinear control systems possessing symmetries," *IEEE Trans. on Automatic Control*, vol. 30, no. 3, pp. 248–258, 1985.
- [4] Y.-W. Jing, G. M. Dimirovski, S.-Y. Zhang, M. J. Stankovski, and T. D. Kolemisevska-Gugulovska, "Stabilizing state-feedback control of electrical power systems interconnected with delay interconnections," in *Proc. ETAI 2000 - the 5th National Conference with International Participation*, The Macedonian IFAC NMO Society ETAI, Skopje, Sept. 2000, pp. A120–A127.
- [5] M. A. Pai, *Power System Stability: Analysis by Direct Method of Lyapunov*. Amsterdam: North-Holland, 1981.

- [6] W. Respondek, "Partial linearization, decomposition and fibre linear systems," in *Theory and Application of Nonlinear Control Systems*, C. I. B. . P. Lindquist, Ed. Amsterdam: Elsevier Science, Publishers B.V. North-Holland, 1986.
- [7] Z.-Y. Shan, Y.-W. Jing, and S.-Y. Zhang, "Similar structure system decomposition for a kind of control systems," in *Proc. of the National Conference on Control and Decision*, Harbin, P.R. of China, 1992, pp. 125–128.
- [8] A. J. Van der Schaft, "Stochastic realization of spectral density matrices which may be possessing symmetries," in *Proc. of the 23rd IEEE CDC Conference*, The IEEE, New York, 1984, pp. 1484–1487.
- [9] D. J. Tanevska, G. M. Dimirovski, M. J. Stankovski, and Y.-W. Jing, "On simulation based analysis and synthesis of hierarchical network structures," in *Proc. of the 6th European Control Conference*, The EUCA, the APCa and Universidade do Porto, Porto, 2001, pp. 797–802.
- [10] S.-Y. Zhang, "The structures of symmetry and similarity of complex control systems," *Control Theory & Applications*, no. 2, pp. 231–237, 1994, (in Chinese).
- [11] C. T. Chen, *Linear System Theory and Design*. New York: New Holt, Reinhart and Winston, 1984.
- [12] G. M. Dimirovski, N. E. Gough, and S. Barnett, "Categories in systems and control theory," *Intl. J. Syst. Sci.*, vol. 8, no. 9, pp. 1081–1090, 1977.
- [13] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer-Verlag, 1995.
- [14] Y.-W. Jing, "Control of interconnected large scale systems," ASE-ETF Institute of SS Cyril & Methodius University, Skopje, ASE Technical Report on Postdoctoral Specialization (supervisor G.M. Dimirovski), Jan. 1997.
- [15] Y.-W. Jing, S.-Y. Zhang, and G. M. Dimirovski, "Analytical design of decentralized stabilization control for large-scale delay-interconnection systems," in *Proc. of the UKACC International Conference on Control, IEE Publication No. 455*, vol. I, The Instn. Electrical Engineers, London, Sept. 1998, pp. 364–369.
- [16] H.-M. Liu, L.-Q. Gao, and S.-Y. Zhang, "On the properties of the hierarchical systems possessing similar structure," in *Proc. of the National Conference on Control and Decision*, Harbin, P.R. of China, 1992, pp. 121–124.
- [17] J. Lunze, "Stability analysis of large-scale systems composed of strongly coupled similar subsystems," *Automatica*, vol. 25, no. 4, pp. 561–570, 1989.