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On the Probability Stability of Discrete-Time Control Systems

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Abstract: The problem of probability stability discrete-time control system is considered. A method for stability estimation of the arbitrary order systems is given. Probability stability discrete-time control systems with random parameters are also analyzed.

Keywords: Continuous system, discrete-time system, probability stability, bilinear transformation, random parameters.

1 Introduction

In chemical industry, plastic industry and especially in rubber industry, there are many systems with stochastic parameters. For instance, processes with plasticity and/or elasticity properties may have random parameters. A well-known fact is that, the value of system parameters is determining the stability of the. In case of the systems with determined parameters, the systems can be stable or unstable. In case of the systems with random parameters, the systems can be stable, unstable or stable with some probability. This paper deals with determined probability stability of the system with random parameters. There are several modes of stochastic stability: stability of probability, stability of the K-th moment, almost certain stability, Lyapunov average stability, exponential stability in the K-th moment, monotonic entropy stability, asymptotic entropy stability etc. For all definitions of stochastic stability it is necessary to for probability stability to be $p = 1$. Caughey and Gray [1, 2] have determined almost certain stability of linear dynamic systems with stochastic coefficients. Khasminski [3], Kozin [4, 5] and Pinsky [6] gave various definitions and properties of stochastic stability of ordinary differential equations.

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Necessary and sufficient conditions guaranteeing the average square stability were obtained by Sawaragi [7]. In researching stochastic stability of systems with random parameters, we have also considered the systems with random time-varying parameters by Gaussian distribution [8]. Stochastic stability of systems with random imperfection is considered in [9, 10]. Probability stability estimation of the linear systems with random parameters is considered in [11] for continuous systems, and in [12, 13] for discrete-time systems. In [14], the geometrical imperfection is interpreted as having spatially fluctuating structural properties with respect to a perfect geometry. In [15], the failure probability of the systems has been considered. This probability is computed by integration of the probability density $f(Y)$ of random variable vector Y over the failure domain F [15]

$$p_f = \int_F f(Y)\mathrm{d}Y. \tag{1}$$

In this paper we considered stability probability of the systems with imperfect parameters. This probability is computed by integration of the probability density of the random parameters over the stability domain S in the parametric space

$$p_s = \int_V f(a)\mathrm{d}a. \tag{2}$$

A method for calculation probability stability of discrete-time control systems is given. In addition, for third order discrete-time control system is exactly determined probability stability. Formulas for estimated probability stability for arbitrary order discrete-time control systems are performed.

2 Determining of Probability Stability of the Control Systems

Let the mathematical model of discrete-time control system be given by difference equation

$$\sum_{i=0}^{n} a_i x(k-i) = u(k-n), a_0 = 1, \tag{3}$$

or transfer function

$$W(z) = \frac{z^{-n}}{1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n}}.$$

The system stability (3) is determined by zero of characteristic equation

$$1 + a_1 z^{-1} + \cdots + a_{n-1} z^{-(n-1)} + a_n z^{-n} = 0. \tag{4}$$

Note that the coefficients of characteristic equation are random variables. In the case when the parameters are constant system (3) is stable or not stable, depending of the values of parameters. In the case when the parameters are random system (3) may be stable with probability stability p. The aim of this paper is determining the probability stability p. It is well known that the system (3) is stable, if all roots of the equation (4) are located inside the unit circle. Using the bilinear transformation, $z^{-1} = \frac{s+1}{s-1}$, and after rearrangement, we obtain new equation

$$\varphi_n s^n + \varphi_{n-1} s^{n-1} + \cdots + \varphi_1 s + \varphi_0 = 0, \tag{5}$$

where φ_i are functions of parameters $a_0, a_1, ..., a_n$. For instance, in the case of the second order system we have

$$1 + a_1 z^{-1} + a_2 z^{-2} = 0, \tag{6}$$

and using $z^{-1} = (s+1)/(s-1)$ we obtain

$$s^2 (1 + a_1 + a_2) + 2s(1 - a_2) + 1 - a_1 + a_2 = 0, \tag{7}$$

then

$$\varphi_0 = 1 - a_1 - a_2, \quad \varphi_1 = 2(1 - a_2), \quad \varphi_2 = 1 - a_1 + a_2. \tag{8}$$

System (3) is stable if the all zeros of (5) are in the left half of s plane. It is well-known that polynomial (5) has all zeros in the left half of s plane, if all diagonal minors of D_i determinant

$$D_n = \begin{vmatrix} \varphi_{n-1} & \varphi_{n-3} & \varphi_{n-5} & \cdots & 0 \\ \varphi_n & \varphi_{n-2} & \varphi_{n-4} & \cdots & 0 \\ 0 & \varphi_{n-1} & \varphi_{n-3} & \cdots & 0 \\ 0 & \varphi_n & \varphi_{n-2} & \cdots & 0 \\ . & . & . & \ddots & . \\ 0 & 0 & 0 & \cdots & \varphi_0 \end{vmatrix}, \tag{9}$$

are positive, i.e.

$$\begin{gathered} D_1 = \varphi_{n-1} > 0, \quad D_2 = \begin{vmatrix} \varphi_{n-1} & \varphi_{n-3} \\ \varphi_n & \varphi_{n-2} \end{vmatrix} > 0, \\ D_2 = \begin{vmatrix} \varphi_{n-1} & \varphi_{n-3} & \varphi_{n-5} \\ \varphi_n & \varphi_{n-2} & \varphi_{n-4} \\ 0 & \varphi_{n-1} & \varphi_{n-3} \end{vmatrix} > 0, \quad ... \quad \text{and} \quad D_n > 0. \end{gathered} \tag{10}$$

Stability region of the system (3) is determined by inequalities (10) in parametric space. For the first order system stability region is given by

$$-1 < a_1 < 1. \tag{11}$$

For the second order system stability region V_2 in the parametric plane a_1, a_2 is given by (8)

$$\begin{aligned} &1 - a_1 + a_2 \geq 0, \\ &1 + a_1 + a_2 \geq 0, \\ &a_2 \leq 1, \end{aligned} \tag{12}$$

as shown in Fig. 1. Using previous method, for the third order system and also

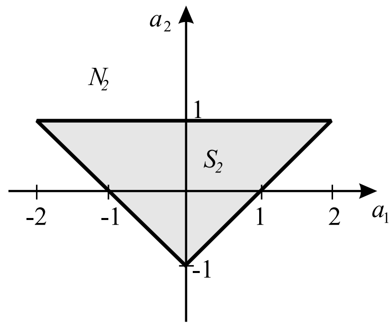

Fig. 1. Stability region - second order system.

using (10) we obtain stability region in the parametric space $(a_1,\ a_2,\ a_3)$

$$\begin{aligned} &a_1 + a_2 + a_3 > -1, \\ &a_1 - a_2 + a_3 < 1, \\ &a_1 a_3 + 1 > a_2 + a_3^2. \end{aligned} \tag{13}$$

The stability region V_3 in the parametric space $(a_1,\ a_2,\ a_3)$ is given in Fig. 2.

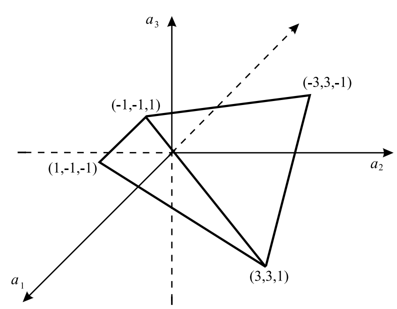

Fig. 2. Stability region - third order system.

For the n-th order systems stability region is also determined by using inequalities (10). Let multidimensional density probability parameters $a_1,\ a_2,\ \ldots,\ a_n$ be

$$f(\boldsymbol{a}) = f(a_1, a_2, \ldots, a_n). \tag{14}$$

Thus probability stability of the system (3) is

$$P = \int_{V_n} \dots \int f(a_1, \dots, a_n) \mathrm{d}a_1 \dots \mathrm{d}a_n. \tag{15}$$

In the case of the n-th order systems, determination of the stability region V_n is very complicated. We can estimate the stability region of the system (3) using the following theorems [12, 9].

Theorem 1 *If*

$$|a_1| + |a_2| + \dots + |a_n| < 1, \tag{16}$$

then all zeros of the polynomial (4) are located within the unit circle, i.e.

$$|z_i| < 1, \quad i = 1, 2, \dots, n. \tag{17}$$

These are necessary but not sufficient conditions for stability of the system (3).

The region $|a_1| + |a_2| + \dots + |a_n| < 1$ in n-th order parametric space is denoted with V_{on}.

Theorem 2 *A Necessary condition for the polynomial (4), is to have all zeros of the polynomial (4) located within the unit circle.*

$$|a_i| \leq \binom{n}{i}, \quad i = 1, 2, \dots, n. \tag{18}$$

Condition (18) is not sufficient, for the stability of the system (3).

The region $\binom{n}{i}$, $i = 1, 2, \dots, n$ in n-th order parametric space is denoted with V_{pn}.

Using the Theorems 1 and 2 we obtain

$$V_{on} \in V_n \in V_{pn}. \tag{19}$$

The probability stability for the n-th order system can be determined by

$$\begin{aligned} p_o &= \int_{V_{on}} \cdots \int f(a_1, \dots, a_n) \mathrm{d}a_1 \cdots \mathrm{d}a_n, \\ p_p &= \int_{V_{pn}} \cdots \int f(a_1, \dots, a_n) \mathrm{d}a_1 \cdots \mathrm{d}a_n, \\ p_o &< p < p_p, \end{aligned} \tag{20}$$

In case of the Gaussian probability density function

$$f_i(a_i) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{(a-\bar{a}_i)^2}{2\sigma_i^2}},$$
$$f(a_1, a_2, ..., a_n) = \prod_{i=1}^{n} f_i(a_i) = \frac{1}{(\sqrt{2\pi})^n \prod_{i=1}^{n} \sigma_i} e^{\sum_{i=1}^{n} \frac{(a-\bar{a}_i)^2}{2\sigma_i^2}}, \tag{21}$$

using (18) and (20) the probability stability can be determined by

$$p_p \approx \left(\frac{1}{2}\right)^n \prod_{i=1}^{n} \left\{ \mathrm{erf}\left[\frac{\binom{3}{i} - \bar{a}_i}{\sqrt{2}\sigma_i}\right] - \mathrm{erf}\left[\frac{-\binom{3}{i} - \bar{a}_i}{\sqrt{2}\sigma_i}\right] \right\}, \tag{22}$$

where erf is the error function.

In the case of the uniform distribution

$$f_i = \begin{cases} \dfrac{1}{a_i^+ - a_i^-}, & \text{for} \quad a_i^- \leq a_i \leq a_i^+ \\ 0, & \text{for} \quad a_i^+ < a_i < a_i^+ \end{cases} \tag{23}$$

Using (18), (20) and (23) we obtain

$$p < p_p = \frac{\prod_{i=1}^{n} \{[|a_i^+ + \binom{n}{i}| - |a_i^+ - \binom{n}{i}|] - [|a_i^- + \binom{n}{i}| - |a_i^- - \binom{n}{i}|]\}}{2^n \prod_{i=1}^{n} (a_i^+ - a_i^-)}. \tag{24}$$

Using (16), (20) and (23) we obtain

$$p > p_o = 1 - \frac{\frac{1}{n}\left[\left(a_1^+ + a_2^+ + ... + a_n^+ - 1\right)^n + ... + \left(a_1^+ + a_2^- + ... + a_n^- - 1\right)^n\right]}{\prod_{i=1}^{n} \left(a_i^+ - a_i^-\right)}. \tag{25}$$

3 Examples

First order system is given by mathematical model

$$x((k+1)T) + a_1 x(kT) = u(kT). \tag{26}$$

For T=1 characteristics polynomial is

$$z + a_1 = 0, \tag{27}$$

where stability domain is $|a_1| < 1$. Let a_1 is random parameter with Gaussian probability function $(\sigma = 1, \bar{a} = 0.7)$

$$f(a_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(a_1-0.7)^2}{2}}, \tag{28}$$

Probability stability (20) is

$$p = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-\frac{(a_1-0.7)^2}{2}} \mathrm{d}a_1 = 0.573. \tag{29}$$

See in Fig.3.

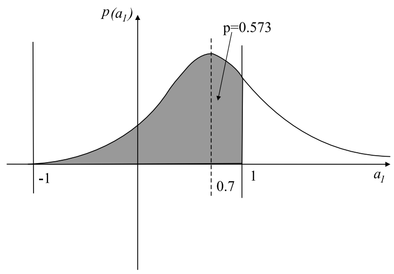

Fig. 3. Probability stability - first order system.

Let us compare our result with stochastic stability. Necessary condition for stability of the system (26) is

$$1 + \sum_{k=1}^{\infty} E\left(a_1^{2k}\right) < \infty. \tag{30}$$

For $\sigma = 1$, $a_1 = 0.7$ we obtain
$E\left(a_1^2\right) = 1.903$, $E\left(a_1^4\right) = 4.851$, $E\left(a_1^6\right) = 18.497$,
$E\left(a_1^8\right) = 92.915$, $E\left(a_1^{10}\right) = 576.341$,
$E\left(a_1^{120}\right) = 4.455 \times 10^7$.

Thus system (26) is unstable. This is not in the contradictory with respect to our result (29), because for stochastic stability it is necessary to be p=1. In our case p=0.573 , i.e. the system maybe stable with probability stability p.

In the case of the uniform distribution

$$f(a_1) = \begin{cases} \dfrac{1}{a_1^+ - a_1^-}, & \text{for} \quad a_1^- \le a_1 \le a_1^+ \\ 0, & \text{for} \quad a_1^+ < a_1 < a_1^- \end{cases} \tag{31}$$

for a_1^- and $a_1^+ > 0$, probability stability is

$$p = \begin{cases} 1, & \text{for} \quad a_1^- < a_1 < 1 \\ \dfrac{1 - a_1^-}{a_1^+ - a_1^-}, & \text{for} \quad a_1^- < 1 < a_1^+ \\ 0, & \text{for} \quad 1 < a_1^- < a_1^+ \end{cases} \tag{32}$$

For stochastic stability in the case of the uniform distribution we have

$$E\left(a_1{}^{2k}\right) = \frac{a_1^{+\,2k+1} - a_1^{-\,2k+1}}{(2k+1)\left(a_1^+ - a_1^-\right)}, \tag{33}$$

it is obvious that the system (30) is stochastic stable when $0 < a_1^- < a_1^+ < 1$ and unstable when $a_1^+ > 1$, that in accordance with (32).

The second order system is given by

$$x((k+2)T) + a_1 x((k+1)T) + a_2 x(kT) = u(kT). \tag{34}$$

For T=1 characteristics polynomial is

$$z^2 + a_1 z + a_2 = 0. \tag{35}$$

Let a_1 and a_2 are random parameters with Gaussian probability density function $(\sigma_1 = 1,\ \bar{a}_1 = 0.7,\ \sigma_2 = 1,\ \bar{a}_2 = 0.7)$

$$f(a_1, a_2) = \frac{1}{2\pi} e^{-\left[\frac{(a_1 - 0.7)^2}{2} + \frac{(a_2 - 0.7)^2}{2}\right]}. \tag{36}$$

Probability stability is

$$p = \frac{1}{2\pi} \int_{-1}^{1} \int_{a_2 - 1}^{a_2 + 1} e^{-\left[\frac{(a_1 - 0.7)^2}{2} + \frac{(a_2 - 0.7)^2}{2}\right]} . \mathrm{d}a_1 \mathrm{d}a_2 = 0.333 \tag{37}$$

See Fig. 4.

Let discrete-time control system where are $a_1,\ a_2,\ a_3$ random parameters with Gaussian probability density function where: $\bar{a}_1 = 0.5,\ \bar{a}_2 = 0.3,\ \bar{a}_3 = 0.2$ and $\sigma_1 = 0.4,\ \sigma_2 = 0.2,\ \sigma_3 = 0.5.$

Mathematical model of the given system is

$$x((k+3)T) + a_1 x((k+2)T) + a_2 x((k+1)T) + a_3 x(kT) = u(kT). \tag{38}$$

For T=1 characteristics polynomial is

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0. \tag{39}$$

Using (13) and (15) we obtain exact value for probability stability: $p = 0.937$.

Using (21), for $n = 3$ estimated probability stability P_p is

$$P_p \approx \frac{1}{8} \prod_{i=1}^{3} \left\{ \mathrm{erf}\left[\frac{\binom{3}{i} - \bar{a}_i}{\sqrt{2}\sigma_i}\right] - \mathrm{erf}\left[\frac{-\binom{3}{i} - \bar{a}_i}{\sqrt{2}\sigma_i}\right] \right\}, \tag{40}$$

i.e. $P_p \approx 0.948$. Thus $p < p_p$, that is expected result.

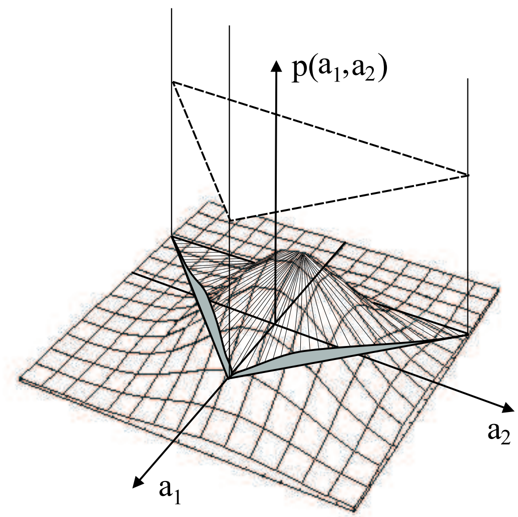

Fig. 4. Probability stability - second order system.

4 Conclusion

A method for probability stability and reliability estimation discrete-time control system with random parameters is presented. As we can see, the exact determination of probability stability is very difficult for the high order systems. The formulas, for estimation probability stability and reliability for high order discrete-time system, are also given.

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