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Sensitivity Optimization of Direct Form Realization of Active-RC All-pole Filters

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Abstract: In this paper, a procedure for direct form realization of Active-RC all-pole filters is presented. The filters are based on the resistance-capacitance ladder structure combined with the single operational amplifier and multiple feedback for complex pole realization. Recursive formulas have been developed that determine transfer function coefficients and derivation of transfer function coefficient with respect to filter components. These design equations are developed without knowing filter coefficients in terms of filter components in closed form, and they are necessary and sufficient for the filter design.

Performance of these filters hase been compared with the classical cascade form realization.

Keywords: Signal processing, active RC filter, direct form realization, sensitivity.

1 Introduction

As it is known, to realize a transfer function in an active RC cascade form, the specified transfer function is broken up into products of the second- and first-order sections [1]. This brings to tree basic step in active filter design, i.e., the decomposition. The degrees of freedom in this decomposition are: the pole-zero pair selection, the gain distribution and the cascading sequence. Criteria for which a filter to be optimized by means of these degrees of freedom are: maximizing of dynamic range, minimizing overall transmission sensitivity, minimizing DC offset, maximizing the signal-to-noise ratio and so on. Other criteria for optimization may be required for the given filter, but one listed above may be considered the most important.

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The proposed direct form realization of the active-RC all-pole filters is without decomposition. This realization is based on the resistance-capacitance ladder structure combined with the single operational amplifier and multiple feedback for complex pole realization [2]. These filter structures are obtained by expanding the all-pole Sallen and Key filter structure which is characterizing positive feedback network and op-amp in the non-inverting mode. Thus, this filter can be used to obtain low-power high-order active RC filters (i.e., only one op-amp per filter) and low cost (i.e., no need analog-to-digital and digital-to-analog converter, anti-aliasing filter, digital hardware, etc).

The design equations for filters of higher than the third-order have been developed. First, recursive formulas that determine the coefficients of the all-pole transfer function, and second, the recursive formulas for determining derivate of the filter coefficient with respect to the values of the filter components. These formulas are necessary and sufficient to find filter components. Finally, component sensitivities of the filter coefficient, ω and Q, are considered.

2 Recursive Formula

Direct form realization of active-RC all-pole filters of n-order shown in Fig. 1(a), and in Fig. 1(b) is its equivalent representation.

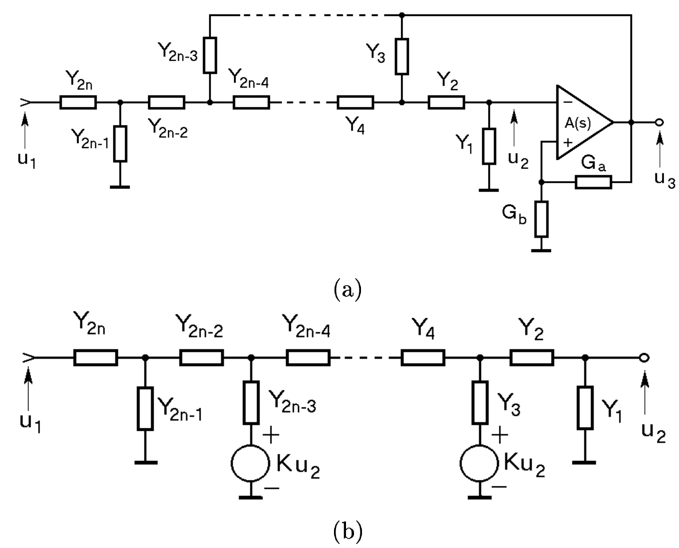

Fig. 1. General nth-order single-amplifier low-pass filter. (a) Network. (b) Equivalent diagram.

This network can be conventionally described by a set of node equations of the form

$$\begin{bmatrix} Y_{n,n} & Y_{n,n-1} & 0 & \cdots & 0 & Y_{n,1} \\ Y_{n-1,n} & Y_{n-1,n-1} & Y_{n-1,n-2} & \cdots & 0 & Y_{n-1,1} \\ 0 & Y_{n-2,n-1} & Y_{n-2,n-2} & \cdots & 0 & Y_{n-2,1} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & Y_{2,2} & Y_{2,1} \\ 0 & 0 & 0 & \cdots & Y_{1,2} & Y_{1,1} \end{bmatrix} \begin{bmatrix} E_{n-1} \\ E_{n-2} \\ \vdots \\ E_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} sU_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{1}$$

where for the low-pass filter structure $Y_{i,i} = G_{2i} + G_{2i-2} + C_{2i-1}s$, $Y_{j,i} = Y_{i,j} = -G_{2i-2}$, $Y_{i,1} = -0.5[(-1)^i + 1]kC_{2i-1}$, except $Y_{2,1} = -(G_2 + kC_3 s)$ with $G_0 = 0$ and $k = 1 + G_b/G_a$ if op-amp is ideal, for $i = 1, 2, \ldots, n$.

Solving (1) for $U_3 = kU_2$ yields

$$U_0 = k\frac{\Delta_{1,n}}{\Delta_n}U_1, \tag{2}$$

where Δ_n is a determinant of the system equation (1) and $\Delta_{1,n}$ is cofactor $(1, n)$ in Δ_n. The determinant Δ_n can be evaluated by the Laplace expression

$$\begin{aligned} \Delta_n =& [G_{2n} + G_{2n-2} + C_{2n-1}s]\Delta_{n-1} - G_{2n-2}^2\Delta_{n-2} \\ & - \left[\frac{(-1)^n + 1}{2}\right]kC_{2n-1}s\prod_{m=1}^{n-1}G_{2m}, \end{aligned} \tag{3}$$

where Δ_{n-1} and Δ_{n-2} are determinants of the system equations for the filter order $n-1$ and $n-2$, respectively. The cofactor $\Delta_{1,n}$ in Δ_n can be evaluated from the relationship

$$\Delta_{1,n} = (-1)^{n+1}\prod_{m=1}^{n}G_{2m}. \tag{4}$$

As a corollary from (3), it is evident that determinant of the system equating Δ_n is a simple function of Δ_{n-1} i Δ_{n-2}. Since

$$\begin{aligned} \Delta_{n-1} &= b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_0 \\ \Delta_{n-2} &= a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + \cdots + a_0 \end{aligned} \tag{5}$$

the substitution of (5) into (3) yields

$$\Delta_n = d_n s^n + d_{n-1}s^{n-1} + \cdots + d_1 s + d_0. \tag{6}$$

There is a recursive formula that determines coefficients d_i's from those of the $(n-1)$th and $(n-2)$th-order polynomials with coefficients b_i's and a_i's

$$\begin{aligned} d_i =&(G_{2n} + G_{2n-2})b_i + C_{2n-1}b_{i-1} - G_{2n-2}^2 a_i \\ &- \delta_i \Big[\frac{(-1)^n + 1}{2}\Big] k C_{2n-1} \prod_{m=1}^{n-1} G_{2m}, \end{aligned} \tag{7}$$

for $i = 0, 1, \ldots, n-1$, where

$$\delta_i = \begin{cases} 1, & \text{for } i = 1 \\ 0, & \text{for } i \neq 1 \end{cases} \tag{8}$$

Initial values are the coefficients of the first-order polynomial $a_1 = C_1$ and $a_0 = G_2$, and coefficients of the second-order polynomial $b_2 = C_1C_3$, $b_1 = (G_4 + G_2)C_1 + (1-k)G_2C_3$ and $b_0 = G_2G_4$.

Recalling two important properties of the coefficients d_i, namely: first, the coefficient d_i must be real and second they must be positive.

3 Design

The design equations, in addition to being nonlinear, contain more unknowns than be solved for explicitly. Assuming all resistor (or capacitors) and gain k are known in the configuration of Fig. 1(b), it is now possible to determine the remaining network components by solving the corresponding set of n nonlinear equations with n unknowns. The coefficients d_i can, for example, be equated to those of the appropriate coefficients α_i of the Butterworth filter or coefficients of other all-pole filters with low passband ripple

$$\boldsymbol{F}(x) = \boldsymbol{f}(d) - \boldsymbol{\alpha} = 0 \tag{9}$$

were $\boldsymbol{\alpha} = [\alpha_0, \alpha_2, \cdots, \alpha_{n-1}]$ is a constant vector, $\boldsymbol{f} = [d_0(\boldsymbol{x}), d_2(\boldsymbol{x}), \ldots, d_{n-1}(\boldsymbol{x})]$ is a vector which is derived by recurrence equation (7) and $\boldsymbol{x} = [G_2, G_4 \ldots, G_{2n}]$ or $\boldsymbol{x} = [C_1, C_3 \ldots, C_{2n-1}]$ is a vector which contains n unknown filter components. Thus, equation (9) constitutes a system of n equations with n unknowns, which can be solved to obtain $[G_2, G_4, \ldots, G_{2n}]$ or $[C_1, C_3, \ldots, C_{2n-1}]$ if other filter components are known.

There exist several iteration procedures for solving such equations, such as the Newton-Raphson method, which provides a quadratic convergence to the solution $\boldsymbol{x} = [x_1, x_2, \ldots, x_k]$

$$\boldsymbol{x}_{n+1} = \boldsymbol{x}_n - \lambda \boldsymbol{J}^{-1}(\boldsymbol{x}_n)\boldsymbol{F}(\boldsymbol{x}_n) \tag{10}$$

where $\boldsymbol{x}_0$ is a vector which contains initial guess of the unknown filter parameter values and $\boldsymbol{J}(x)$ is the Jacobian matrix of $\boldsymbol{F}(x)$

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial d_0}{\partial x_1} & \frac{\partial d_0}{\partial x_2} & \cdots & \frac{\partial d_0}{\partial x_n} \\ \frac{\partial d_1}{\partial x_1} & \frac{\partial d_1}{\partial x_2} & \cdots & \frac{\partial d_1}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial d_{n-1}}{\partial x_1} & \frac{\partial d_{n-1}}{\partial x_2} & \cdots & \frac{\partial d_{n-1}}{\partial x_n} \end{bmatrix}, \tag{11}$$

and $0 < \lambda \leq 1$ is a parameter which may improve the convergence. The recurrence relations for $\partial d_i / \partial x_j$, $i = 0, 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n$ are derived from (7). The closed form of the analytical equations of these derivations are not necessary. Only numerical values are sufficient for solving the system equation (9).

The recurrence formulas for computing partial derivates of d_i with respect to each conductance are

$$\begin{aligned} \frac{\partial d_i}{\partial G_{2j}} =&(G_{2n} + G_{2n-2}) \frac{\partial b_i}{\partial G_{2j}} + C_{2n-1} \frac{\partial b_{i-1}}{\partial G_{2j}} - G_{2n-2}^2 \frac{\partial a_i}{\partial G_{2j}} \\ &- \delta_i \Big[\frac{(-1)^n + 1}{2} \Big] k C_{2n-1} \prod_{\substack{m=1 \\ m \neq i}}^{n-1} G_{2m}, \\ \frac{\partial d_i}{\partial G_{2n-2}} =&b_i + (G_{2n} + G_{2n-2}) \frac{\partial b_i}{\partial G_{2n-2}} + C_{2n-1} \frac{\partial b_{i-1}}{\partial G_{2n-2}} - 2G_{2n-2} a_i \\ &- \delta_i \Big[\frac{(-1)^n + 1}{2} \Big] k C_{2n-1} \prod_{m=1}^{n-2} G_{2m} \\ \frac{\partial d_i}{\partial G_{2n}} =&b_i. \end{aligned} \tag{12}$$

for $j = 1, 2, \ldots, n-2$ and $i = 0, 1, 2, \ldots, n-1$. These formulas are useful if the numerical values for capacitance are known.

The recurrence formulas for computing partial derivates of d_i with respect to each capacitance are

$$\begin{aligned} \frac{\partial d_i}{\partial C_{2j-1}} =&(G_{2n} + G_{2n-2}) \frac{\partial b_i}{\partial C_{2j-1}} + C_{2n-1} \frac{\partial b_{i-1}}{\partial C_{2j-1}} - G_{2n-2}^2 \frac{\partial a_i}{C_{2j-1}} \\ \frac{\partial d_i}{\partial C_{2n-1}} =& b_{i-1} - \delta_i \Big[\frac{(-1)^n + 1}{2} \Big] k \prod_{m=1}^{n-1} G_{2m} \end{aligned} \tag{13}$$

for $j = 1, 2, \ldots, n-1$ and $i = 0, 1, 2, \ldots, n-1$. These formulas are useful if the numerical values for resistances are known.

Finally, the recurrence formulas for computing partial derivates of d_i with respect to gain k are

$$\begin{aligned}\frac{\partial d_i}{\partial k} =&(G_{2n} + G_{2n-2})\frac{\partial b_i}{\partial k} + C_{2n-1}\frac{\partial b_{i-1}}{\partial k} - G_{2n-2}^2\frac{\partial a_i}{\partial k} \\ &- \delta_i\Big[\frac{(-1)^n + 1}{2}\Big] C_{2n-1}\prod_{m=1}^{n-1} G_{2m}\end{aligned} \tag{14}$$

for $i = 0, 1, 2, \ldots, n-1$. These formulas are suitable for computing sensitivity of filter coefficients with respect to gain k.

As initial values partial derivates of coefficients of the first-order polynomial a_i and partial derivates of coefficients of the second order polynomial b_i with respect to components G_j, C_j and k, are used.

Vector $\boldsymbol{x}_0$, which contains normalized initial guess of filter parameters, $\boldsymbol{x}_0 = [1, 1, \ldots, 1]$ is a good choice for equal capacitances, or equal resistances, low passband ripple filter.

The coefficients d_i can be, for example, equated to the five-order Chebyshev filter with 0.5 dB passband ripple, resulting in equal-valued capacitor filter ($C_1 = \cdots = C_9 = 1$) with the following normalized components $G_2 = 9.3272$, $G_4 = 0.6609$, $G_6 = 0.8755$, $G_8 = 0.9129$, $G_{10} = 0.2030$ and $k = 2.2$.

4 Sensitivity Consideration

In this section, the coefficient sensitivity and sensitivity of the parameters ω_p and Q_p of the n-th order all-pole filters to any element of the network without analytical expressions for coefficient sensitivity to the network elements are comuted. It is shown that the method introducing the filter sensitivity reduction is effective in reduction of the coefficients or pole frequency ω_p and pole Q_p variations due to the components tolerances.

4.1 Coefficient sensitivity

The relative sensitivity of a coefficient d_i to variation of components x (G_j, C_j or k) is defined as

$$\mathcal{S}_x^{d_i} = \frac{x}{d_i}\frac{\partial d_i}{\partial x}. \tag{15}$$

Substituting (12), (13) and (14) in (15) these sensitivities can be computed and relative change of d_i to the variation components obtained. For the fifth order equal-valued capacitor Butterworth filter ($C_1 = \cdots = C_9 = 1$) relative coefficient sensitivities to the element change are listed in Table 1. Normalized values of resistors conductivity of this filter are: $G_2 = 9.3272$, $G_4 = 0.66087$, $G_6 = 0.87554$, $G_8 = 0.91285$ and $G_{10} = 0.20298$.

Table 1. Coefficients sensitivities of the fifth-order Butterworth filter with equal capacitors and $k = 2.2$.

$\mathcal{S}_x^{d_i}$	d_0	d_1	d_2	d_3	d_4
G_2	1	0.9669	0.4514	−0.4048	−0.5765
G_4	1	1.0935	2.0159	1.7854	0.4084
G_6	1	0.7176	−0.8914	0.2335	0.5411
G_8	1	1.1354	1.2805	0.3037	0.5642
G_{10}	1	0.0866	0.1436	0.0822	0.0672
C_1	−1	1.7146	8.2019	8.3151	3.4587
C_3	−1	−4.2177	−11.2063	−9.5646	−3.0865
C_5	−1	1.2139	2.1516	−0.2689	-0.4748
C_7	−1	−3.2331	−1.4107	−0.1889	−0.5526
C_9	−1	0.5224	−0.7364	−0.2927	−0.3448
k	0	−9.9932	−21.3048	−17.4026	−6.3410

Before interpreting these results, it is useful to point out their adherence to the property of the coefficient sensitivity invariant as given for the fifth-order network with

$$\sum_{\mu=1}^{5} \mathcal{S}_{G_{2\mu}}^{d_{5-i}} = -\sum_{\mu=1}^{5} \mathcal{S}_{C_{2\mu-1}}^{d_{5-i}} = i. \qquad (16)$$

for $i = 1, 2, \ldots, 5$. Equation(16) is suitable for verification the results which are given in Tables 1 and 2, and it is satisfied for our example.

As shown in Table 1, the contribution to the overall sensitivity from the passive components is negligibly small, except of the coefficient sensitivity to the capacitance C_3, but it is much smaller in comparison with coefficient sensitivity to the active element k. The major problem associated with the direct form realization is reduction of the sensitivity to the amplifier gain k.

Coefficient sensitivities are maximal with respect to the gain of amplifier k, and these sensitivities decrease if k decreases. Value of $k = 2.2$ for our fifth-order Butterworth filter is minimum for the filter with the equal capacitor. Values of k less than 2.2 lead toward increasing of sensitivities and to the negative value resistors

Further decreasing the coefficients sensitivities by decreasing gain k can be used for the filter design by impedance scaling upwards from the last G_2C_1-section of the ladder structure.Thus, voltage at the input of the operational amplifier is increased and loading of the next to the last section, decreased In general, for an nth-order filter, as shown in Fig.1, the coefficient sensitivity can be reduced with increasing impedance (decreasing capacitance) of the known elements from the left to the right by the factor ρ^{n-i}, where ρ is a scaling factor. In other word, capacitors C_{2i-1} are chosen in the following manner: $C_{2i-1} = 1/\rho^{n-i}$, for $i = 1, \dots, n$.

In the next example, the capacitor values are optimized for sensitivity reduction of yhe fifth-order Butterworth filter coefficient to the active filter element k. For this filter $\omega_{p_1} = \omega_{p_2} = \sigma_p = 1$, $Q_{p_2} = 0.61803$ and dominant pole Q-factor $Q_{p_1} = 1.618044$. Proposed normalized values for capacitor, which are scaled from the left to the right with scaling factor $\rho = 10$, are: $C_1 = 0.0001$, $C_3 = 0.001$ $C_5 = 0.01$, $C_7 = 0.1$, $C_9 = 1$ and $k = 1.3$. Normalized values for other filter elements, it was determined by the design equation, are: $G_2 = 0.0324$, $G_4 = 0.2347$, $G_6 = 0.2519$ and $G_8 = 0.5543$. Corresponded coefficients sensitivities are listed in Table 2.

Table 2. Coefficients sensitivities of the fifth order Butterworth filter with optimized capacitors ($\rho = 10$) and $k = 1.5$.

$S_x^{d_i}$	d_0	d_1	d_2	d_3	d_4
G_2	1	0.8395	0.4490	−0.0354	−0.2379
G_4	1	1.1149	1.2222	1.0066	0.3657
G_6	1	0.4442	0.0799	0.2710	0.1814
G_8	1	1.0847	0.8306	0.4677	0.5012
G_{10}	1	0.5167	0.4183	0.2900	0.1896
C_1	−1	−0.8047	−0.3613	0.1431	0.2974
C_3	−1	−1.1739	−1.3267	−1.0831	−0.3919
C_5	−1	−0.3951	−0.0396	−0.2903	−0.1982
C_7	−1	−1.1300	−0.7929	−0.4515	−0.4721
C_9	−1	−0.4964	−0.4795	−0.3182	−0.2352
k	0	−0.9115	−1.6317	−1.6158	−0.8922

If the coefficients sensitivities displayed in Tables 1 and 2 are compared, it can be seen that maximal value of the sensitivity coefficient $\mathcal{S}_k^{d_2} = -21.3048$ is reduced to the value $\mathcal{S}_k^{d_2} = -1.6317$ which is about 13 times lower. The coefficient sensitivities to the capacitor C_3 are also much reduced to the value the modulo of which is less than 1.33. Gain $k = 1.5$ is optimum for this design. For k lower or higher than 1.5, sensitivity of coefficient d_i to the gain k increases. In general, a $\rho = 10$ is sufficient to provide a significant degree to insensitivity to all component change.

Note that sensitivity of the filter coefficient d_i decreases if ρ increases. However, decrease in the sensitivity is much lower than increase in of C_{2n-1}/C_1. In practice, a large ρ may cause C_1 to fall into the range of the parasitic capacitance. A conclusion can be made that only great rate $C_{2n-1}/C_1 = \rho^{n-1}$ is disadvantage of the direct form realization in comparison with the cascade form realization.

Since pole frequencies and Q-sensitivities are more significant for the filter sensitivities than the coefficient sensitivities, these sensitivities will be computed in the next section for direct form realization of the fifth-order filter.

4.2 Sensitivity of Q and ω

The relationship between the coefficient sensitivity and the ω and Q sensitivities can readily be obtained. Consider the denominator of fifth-order transfer function

$$D(s) = d_5 s^5 + d_4 s^4 + d_3 s^3 + d_2 s^2 + d_1 s + d_0. \tag{17}$$

Assuming, for simplicity, that $d_5 = 1$ because d_5 can be included in the other coefficient and in the multiplicative constant of the transfer function, $D(s)$ can be written in the terms of the parameters ω_{p_1}, ω_{p_2}, Q_{p_1}, Q_{p_2} and σ_p, as

$$D(s) = (s^2 + \frac{\omega_{p_1}}{Q_{p_1}} s + \omega_{p_1}^2)(s^2 + \frac{\omega_{p_2}}{Q_{p_2}} s + \omega_{p_2}^2)(s + \sigma_p) \tag{18}$$

where ω_{p_1}, ω_{p_2} and σ_p are pole frequencies, and Q_{p_1} and Q_{p_2} are complex pole Q. Comparing equation (17) with equation (18) after multiplication, the coefficients d_0, d_1, d_2, d_3 and d_4 are indentified as

$$\begin{aligned}
d_0 =& \omega_{p_1}^2 \omega_{p_2}^2 \sigma_p \\
d_1 =& \omega_{p_1}^2 \omega_{p_2}^2 + \Big(\frac{\omega_{p_1}\omega_{p_2}^2}{Q_{p_1}} + \frac{\omega_{p_1}^2\omega_{p_2}}{Q_{p_2}}\Big)\sigma_p \\
d_2 =& \frac{\omega_{p_1}\omega_{p_2}^2}{Q_{p_1}} + \frac{\omega_{p_1}^2\omega_{p_2}}{Q_{p_2}} + \Big(\omega_{p_2}^2 + \frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}} + \omega_{p_1}^2\Big)\sigma_p \\
d_3 =& \omega_{p_1}^2 + \frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}} + \omega_{p_2}^2 + \Big(\frac{\omega_{p_1}}{Q_{p_1}} + \frac{\omega_{p_1}}{Q_{p_2}}\Big)\sigma_p \\
d_4 =& \frac{\omega_{p_1}}{Q_{p_1}} + \frac{\omega_{p_1}}{Q_{p_2}} + \sigma_p
\end{aligned}$$

The coefficient sensitivities to an element x can then be related to the sensitivities of parameters Q_{p_1}, Q_{p_2}, ω_{p_1}, ω_{p_2} and σ_p to the same element x as follows.

The sensitivity of coefficient d_0 with respect to ω and Q-sensitivities

$$\mathcal{S}_x^{d_0} = 2\mathcal{S}_x^{\omega_{p_1}} + 2\mathcal{S}_x^{\omega_{p_2}} + \mathcal{S}_x^{\sigma_p} \tag{19}$$

The sensitivity of coefficient d_1 with respect to ω and Q-sensitivities

$$\begin{aligned}\mathcal{S}_x^{d_1} =&\Big(2\frac{\omega_{p_1}^2\omega_{p_2}^2}{d_1} + \frac{\omega_{p_1}^2\omega_{p_2}\sigma_p}{Q_{p_1}d_1} + 2\frac{\omega_{p_1}\omega_{p_2}^2\sigma_p}{Q_{p_2}d_1}\Big)\mathcal{S}_x^{\omega_{p_1}}\\ &+\Big(2\frac{\omega_{p_1}^2\omega_{p_2}^2}{d_1} + 2\frac{\omega_{p_1}\omega_{p_2}^2\sigma_p}{Q_{p_1}d_1} + \frac{\omega_{p_1}^2\omega_{p_2}\sigma_p}{Q_{p_2}d_1}\Big)\mathcal{S}_x^{\omega_{p_2}}\\ &-\frac{\omega_{p_1}\omega_{p_2}^2\sigma_p}{Q_{p_1}d_1}\mathcal{S}_x^{Q_{p_1}} - \frac{\omega_{p_1}^2\omega_{p_2}\sigma_p}{Q_{p_1}d_1}\mathcal{S}_x^{Q_{p_2}} + \Big(\frac{\omega_{p_1}\omega_{p_2}^2\sigma_p}{Q_{p_1}d_1} + \frac{\omega_{p_1}^2\omega_{p_2}\sigma_p}{Q_{p_2}d_1}\Big)\mathcal{S}_x^{\sigma_p}\end{aligned} \tag{20}$$

The Sensitivity of coefficient d_2 with respect to ω and Q-sensitivities

$$\begin{aligned}\mathcal{S}_x^{d_2} =&\Big(\frac{\omega_{p_1}\omega_{p_2}^2}{Q_{p_1}d_2} + 2\frac{\omega_{p_1}^2\omega_{p_2}}{Q_{p_2}d_2} + \frac{\omega_{p_1}\omega_{p_2}\sigma_p}{Q_{p_1}Q_{p_2}d_2} + 2\frac{\omega_{p_1}^2\sigma_p}{d_2}\Big)\mathcal{S}_x^{\omega_{p_1}}\\ &+\Big(2\frac{\omega_{p_2}^2\omega_{p_1}}{Q_{p_1}d_2} + \frac{\omega_{p_1}^2\omega_{p_2}}{Q_{p_2}d_2} + \frac{\omega_{p_1}\omega_{p_2}\sigma_p}{Q_{p_1}Q_{p_2}d_2} + 2\frac{\omega_{p_2}^2\sigma_p}{d_2}\Big)\mathcal{S}_x^{\omega_{p_2}}\\ &-\Big(\frac{\omega_{p_1}\omega_{p_2}^2}{Q_{p_1}d_2} + \frac{\omega_{p_1}\omega_{p_2}\sigma_p}{Q_{p_1}Q_{p_2}d_2}\Big)\mathcal{S}_x^{Q_{p_1}} - \Big(\frac{\omega_{p_1}^2\omega_{p_2}}{Q_{p_2}d_2} + \frac{\omega_{p_1}\omega_{p_2}\sigma_p}{Q_{p_1}Q_{p_2}d_2}\Big)\mathcal{S}_x^{Q_{p_2}}\\ &+\Big(\frac{\omega_{p_2}^2}{d_2} + \frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}d_2} + \frac{\omega_{p_1}^2}{d_2}\Big)\sigma_p\mathcal{S}_x^{\sigma_p}\end{aligned} \tag{21}$$

The sensitivity of coefficient d_3 with respect to ω and Q-sensitivities

$$\begin{aligned}\mathcal{S}_x^{d_3} =&\Big(2\frac{\omega_{p_1}^2}{d_3} + \frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}d_3} + \frac{\omega_{p_1}\sigma_p}{Q_{p_1}d_3}\Big)\mathcal{S}_x^{\omega_{p_1}}\\ &+\Big(2\frac{\omega_{p_2}^2}{d_3} + \frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}d_3} + \frac{\omega_{p_2}\sigma_p}{Q_{p_2}d_3}\Big)\mathcal{S}_x^{\omega_{p_2}}\\ &-\Big(\frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}d_3} + \frac{\omega_{p_1}\sigma_p}{Q_{p_1}d_3}\Big)\mathcal{S}_x^{Q_{p_1}} - \Big(\frac{\omega_{p_1}\omega_{p_2}}{Q_{p_1}Q_{p_2}d_3} + \frac{\omega_{p_2}\sigma_p}{Q_{p_2}d_3}\Big)\mathcal{S}_x^{Q_{p_2}}\\ &+\Big(\frac{\omega_{p_1}\sigma_p}{Q_{p_1}d_3} + \frac{\omega_{p_2}\sigma_p}{Q_{p_2}d_3}\Big)\mathcal{S}_x^{\sigma_p}\end{aligned} \tag{22}$$

The sensitivity of coefficient d_4 with respect to ω and Q-sensitivities

$$\mathcal{S}_x^{d_4} = \frac{\omega_{p_1}}{Q_{p_1}d_4}\mathcal{S}_x^{\omega_{p_1}} + \frac{\omega_{p_2}}{Q_{p_2}d_4}\mathcal{S}_x^{\omega_{p_2}} - \frac{\omega_{p_1}}{Q_{p_1}d_4}\mathcal{S}_x^{Q_{p_1}} - \frac{\omega_{p_2}}{Q_{p_2}d_4}\mathcal{S}_x^{Q_{p_2}} + \frac{\sigma_p}{d_4}\mathcal{S}_x^{\sigma_p} \tag{23}$$

Equations (19), (20), (21), (22) and (23) can be put in matrix notation

$$\begin{bmatrix} \mathcal{S}_x^{d_0} \\ \mathcal{S}_x^{d_1} \\ \mathcal{S}_x^{d_2} \\ \mathcal{S}_x^{d_3} \\ \mathcal{S}_x^{d_4} \end{bmatrix} = [S] \begin{bmatrix} \mathcal{S}_x^{\omega_{p1}} \\ \mathcal{S}_x^{\omega_{p2}} \\ \mathcal{S}_x^{Q_{p1}} \\ \mathcal{S}_x^{Q_{p2}} \\ \mathcal{S}_x^{\sigma_p} \end{bmatrix} \tag{24}$$

Solving equation (24) for $\mathcal{S}_x^{\omega_{p1}}$, $\mathcal{S}_x^{\omega_{p2}}$, $\mathcal{S}_x^{Q_{p1}}$, $\mathcal{S}_x^{Q_{p2}}$ and $\mathcal{S}_x^{\sigma_p}$ these are obtainedas a function of coefficient sensitivities $\mathcal{S}_x^{d_0}$, $\mathcal{S}_x^{d_1}$, $\mathcal{S}_x^{d_2}$, $\mathcal{S}_x^{d_3}$ and $\mathcal{S}_x^{d_4}$ which are listed in Tables 1 and 2 for the fifth-order Butterworth filter.

The sensitivity of ω_{p1}, ω_{p1}, Q_{p1}, Q_{p2} and $\mathcal{S}_x^{\sigma_p}$ to the element values of the fifth-order Butterworth filter with equal-valued capacitors and $k = 2.2$, are listed in Table 3. From Table 3 one can see that the component sensitivity to dominant pole Q_{p1} to be very high, especially for gain k (118.0348).

The results for sensitivities, where each capacitor C_{2i-1} is decrease by factor $\rho = 10$, shown in Table 4. Observe that the magnitude of the dominant pole Q_{p1} sensitivities to the gain k is much lower than in the equal capacitor design. It is about 17.5 times lower.

Table 3. ω and Q sensitivities of the fifth-order Butterworth filter with equal capacitors and $k = 2.2$.

$\mathcal{S}_x^{\omega_{pi},Q_{pi}}$	ω_{p1}	ω_{p2}	Q_{p1}	Q_{p2}	σ_p
G_2	1.8930	−1.9295	1.3939	0.0772	1.0729
G_4	1.9810	−5.8832	−13.5084	4.6578	8.8045
G_6	−4.2585	14.5371	12.0529	−4.8626	−19.5572
G_8	1.1945	−3.5200	1.5265	−1.2826	5.6511
G_{10}	0.1900	−2.2043	−1.4649	1.41102	5.0287
C_1	5.8308	−26.8664	−62.3143	17.6287	41.0712
C_3	−11.2770	37.6266	75.1761	−53.6992	−53.6992
C_5	5.8461	−12.3803	−4.9746	12.0683	12.0683
C_7	0.2446	−10.6214	−16.6635	19.7537	19.7537
C_9	−1.6445	11.2416	8.7764	−20.1940	−20.1940
k	−19.4281	54.8208	118.0348	−28.7511	−70.7849

A practical disadvantage of this design is that it requires capacitors of widely differing values ($C_7/C_1 = \rho^4$).

An important property, the so-called dimensional homogeneity property [3] of all active RC filters, also for the direct form realization of the fifth-order filter, is well known which is

$$\begin{aligned} \sum_{\mu=1}^{5} \mathcal{S}_{G_{2\mu}}^{Q_{p_i}} = \sum_{\mu=1}^{5} \mathcal{S}_{C_{2\mu-1}}^{Q_{p_i}} &= 0 \\ \sum_{\mu=1}^{5} \mathcal{S}_{G_{2\mu}}^{\omega_{p_i}} = -\sum_{\mu=1}^{5} \mathcal{S}_{C_{2\mu-1}}^{\omega_{p_i}} &= 1, \end{aligned} \tag{25}$$

where the summation is over all the resistors (capacitors). The relations (25) are valid for both parameters ω_p and Q_p, and they are useful for verification of values in Tables 3 and 4.

Table 4. ω and Q sensitivities of the fifth-order Butterworth filter with optimized capacitors ($\rho = 10$) and $k = 1.5$.

$\mathcal{S}_x^{\omega_{pi},Q_{pi}}$	ω_{p1}	ω_{p2}	Q_{p1}	Q_{p2}	σ_p
G_2	1.0016	−0.6778	0.6858	0.1365	0.3526
G_4	0.6786	−0.8996	−4.0246	1.0567	1.4420
G_6	−0.7287	2.1317	1.2464	−1018	−1.8060
G_8	−0.0491	1.0314	2.4488	−1.5212	−0.9645
G_{10}	0.0976	−0.5856	−0.3563	0.4298	1.9759
C_1	−1.0143	0.6191	−1.1955	−0.361	−0.2098
C_3	−0.7827	1.1376	4.5119	−1.1576	−1.7097
C_5	0.8443	−2.2707	−1.1387	0.0282	1.8528
C_7	0.1246	−1.6194	−2.8979	1.7090	1.9896
C_9	−0.1719	1.1333	0.7201	−0.5435	−2.9230
k	−0.3175	1.0669	6.7389	−0.7703	−1.4988

If Table 3 is compared with Table 4, it can be seen that the dominant pole sensitivity is significantly decreased. The sensitivity of the dominant pole Q-factor to the gain k is decreased from 118.0348 to 6.738, the reduction being about 17.5 times. Other sensitivities are also decreased.

5 Conclusion

A procedure for the all-pole filter design, named direct form realization, has been presented. The filter based on the RC ladder structures is combined with the single operational amplifier. The filter amplifiers provide a low-output impedance and supply positive feedback in order complex conjugate poles to be obtained. Detailed design equations for the nth order low-pass filters are given. Equi-valued resistor and capacitors filter can be accomplished by the iterative procedure.

The proposed direct form realization has improved performance when compared to the cascade form realization. Maximum dynamic range, minimum DC offset, minimum power and maximum signal-to-noise ratios are obtained without optimization. However, the component sensitivity increases with the filter order and inband ripple. Simple method for reducing these sensitivities is also presented. Thus, the direct form realization is very suitable for the maximally flat and low-ripple passband amplitude response filters.

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