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Homoclinic and Heteroclinic Bifurcations in a Two-Dimensional Endomorphism

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Abstract: Our study concerns global bifurcations occuring in noninvertible maps, it consists to show that there exists a link between contact bifurcations of a chaotic attractor and homoclinic bifurcations of a saddle point or a saddle cycle being on the boundary of the chaotic attractor. We provide specific information about the intricate dynamics near such points. We study particularly a two-dimensional endomorphism of $(Z_1 - Z_3 - Z_1)$ type. We will show that points of contact, between boundary of the attractor and its basin of attraction, converge toward the saddle point or the saddle cycle. These points of contact are also points of intersection between the stable and unstable invariant manifolds. This gives rise to the birth of homoclinic orbits (homoclinic bifurcations).

Keywords: Signal processing, homoclinic points, critical curves, bifurcations in endomorphisms, chaos.

1 Introduction

The critical curve notion is an important mathematical tool used to study bifurcations, that take place in invariant areas of two-dimensional endomorphisms, for either invariant absorbing areas or chaotic areas. To our knowledge the notion of critical curve was first introduced in 1964 by Mira [5] who provides an entry into certain areas of current research on noninvertible maps and the role of such curve in bifurcations basin. It is a natural generalization in IR^2 of the notion of critical points of one-dimensional endomorphisms. We define the critical curve LC of an endomorphism T in

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the plane IR^2 (in means of Mira [5]) as the geometrical locus of points x having at least two coincident preimages of first rank. One determines this locus denoted by LC_{-1}, when T is differentiable, by taking the Jacobian of T equal to zero ($J = \det(DT(x,y)) = 0$). A critical line LC is constitute of one or several branches. These branches separate the plane in open regions, where all points of a region have the same number of first rank antecedents.

Since several papers have shown the importance of critical curves in the bifurcations of transition type " *simply connected basin↔nonconnected basin*" as Gumowski and Mira [6] who have developped the role of critical curves in bifurcations, Barugola and Cathala [2,3] and Gardini [4] have studied bifurcations of type " *simply connected basin↔multiply connected basin*". These basic bifurcations result from *the contact of a basin boundary with a critical curve segment of an attracting set*, such a bifurcation leads either to the chaotic area destruction, or a sudden and important modification of the area. Many of chaotic motions that are observed in dynamical systems are intimately associated with the presence of transversal homoclinic points of maps. Contact bifurcations may correspond to homoclinic and heteroclinic bifurcations , and critical curves are useful for interpreting such problems. Bifurcations by homoclinic and heteroclinic contact have been presented in [6,7] for the one-dimensional case, in [4,5] it is proved that some contact bifurcations correspond to homoclinic bifurcations in the case involving a repelling node or focus and other examples of homoclinic orbits of saddles.

It is worth noting that the results presented in this paper were essentially obtained via a numerical way and using the critical curve tool. Unfortunately taking into account the complexity of the matter and its nature and unraveling the dynamics of specific equations often turns out to be analytically insurmountable, it seems then difficult to carry out the study with success from another process.

2 Definitions and Fundamental Properties

In this paragraph, we give the main definitions and properties in report with some technical terms as absorbing area, chaotic area, contact and homoclinic bifurcations .

The endomorphism T considered here defines a discrete dynamical system in R^2

$$(x_{n+1},\ y_{n+1}) = T(x_n, y_n) = (f(x_n, y_n; \lambda), g(x_n, y_n; \lambda))$$

where $f(x,y,\lambda)$ et $g(x,y,\lambda)$ are continuous and differentiable functions with respect to real variables x, y and continuous with respect to the real parameter λ.

Definition 2.1: An absorbing area E is a closed and bounded subset as:

(i) $T(E) \subseteq E$

(ii) its frontier, ∂E is made up of a finite or infinite number of critical arcs of LC, LC_1, LC_2, ..., LC_k, such that $LC = T(LC_{-1})$; $LC_i = T^i(LC)$ for $i \geq 1$.

(iii) A neighborhood $U(E)$ exists, such that preimages of finite rank of its points are in interior of E (all points of U enter into E after a finite number of iterations and cannot get away after entering).

An absorbing area may contain an attracting set.

Definition 2.2: A chaotic area A, is an invariant absorbing area ($T(A) = A$)), the points of which give rise to iterated sequences having the property of sensitivity to initial conditions.

About chaotic areas, it is important to emphasise that te study of such area has only the purpose to obtain properties giving rise to fractal basin boundaries.

Definition 2.3: We say that $\lambda = \lambda^*$ is a bifurcation of contact of E, if a contact between the frontier of E and the frontier its basin of attraction takes place.

Proposition 2.1: *When a bifurcation of contact of a chaotic area A arises for a value $\lambda = \lambda^*$, the crossing of this value leads to the destruction of A,either to a qualitative modification of properties of A (i.e. a sudden and important modification of the size of such an area or its basin of attraction).*

The destruction of A, after the crossing of the bifurcation value has been demonstrated by Gumowski and Mira [6]. The qualitative change of properties of A has been described by Barugola and al [1].

Definition 2.4: Let S be a saddle fixed point of T ; a q point is called homocline to S, so $q \in W^s(S) \cap W^u(S)$ and $q \neq S$. q is a transversal homocline point, so $W^s(S)$ intersects transversally $W^u(S)$.

Definition 2.5: One calls homoclinic orbit $O_o(q)$ associated with q, q belonging to a $U(S)$ of S, a set constituted of successive iterates of q, and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(S)$.

$O_o(q) = \left\{T_l^{-n}(q), q, T^n(q); n > 0\right\} = \{..., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$

where $q_n = T^n(q) \to S$, and $q_{-n} = T_l^{-n}(q) \to S$.

Definition 2.6: One calls heteroclinic orbit $\varepsilon(q)$ connecting S to $S^{'}$ associated with q,the one given by q together with its finite orbit and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of T in $U(S)$.

$\varepsilon(q) = \{T_l^{-n}(q), q, T^n(q); n > 0\} = \{...., q_{-n}, ..., q_{-2}, q_{-1}, q, q_1, q_2, ..., q_n, ...\}$ where $q_n = T^n(q) \to S'$, and $q_{-n} = T_l^{-n}(q) \to S$.

Remark: There is an infinity of homoclinic orbits associated to a homocline point. They have the half positive trajectory, but differ by their half negative trajectory. A homoclinic or heteroclinic orbit is called degenerate if it contains one point of J and nondegenerate otherwise.

Definition 2.7: Let T be an endomorphism of IR^2 depending on a parameter λ and let S be a saddle point of T. A homoclinic bifurcation takes place, if for a value $\lambda = \lambda^*$, there is apparition (or disappearance) of an infinity of homoclinic orbits.

Overlapping of global stable and unstable manifolds usually leads to nonlocal bifurcations. The existence of a transversal point of a planar map leads to very complicated behavior of orbits nearby. Such dynamical complexity is often dubbed as chaos, the onset of chaos typically occurs at the parameters values for which the stable and unstable manifolds of a saddle point come into contact tangentially.

And we have then the

Theorem 2.1 [5]: *Let S be a fixed point of a map T , $T(S) = S$. Let q be a point homoclinic to S, $q \in U(S)$, $U(S)$ being a neighborhood of S such that all the eigenvalues of $DT(x)$ are greater than 1 in absolute value, $\forall x \in U(S)$ and $T(U(S)) \supset U(S)$. If $O_o(q)$ is noncritical ($q_j \in J \backslash LC_{-1}$ for some $1 \leq j \leq (m-1)$) then:*

(a) a set Λ invariant by T^m , for a suitable positive integer m , exists. Λ has a Cantor-like stucture and contains cycles of T^m of any period $p, p \geq 1$;

(b) for any $i \geq 1$, a set Λ invariant by T^{m+i} exists, it has a Cantor-like stucture and contains cycles of T^{m+i} of any period $p, p \geq 1$;

(c) if $O_o(q)$ is nondegenerate homoclinic orbit, then Λ is a repelling set (there exists an integer $N, N \geq 1$,such that abs$J > 1$ for any $x \in \Lambda$ and any $n \geq N$).

Proof: it follows from a construction process using invariance and that all eigenvalues of $DT(x)$ are greater than 1 in absolute value.

Remark: The same role of a homoclinic orbit is also played by a heteroclinic cycle between two fixed points S and S'. That is, results similar to those stated in Theorem 2.1 for a homoclinic orbit of S, can be stated for a heteroclinic cycle, degenerate or nondegenerate.

3 Basin Bifurcations

These bifurcations essentially correspond to an interaction of stable manifold associated with a saddle point or a saddle cycle (they generally constitute the boundary of the basin of attraction of an area or a chaotic attractor) with the critical Lines (which constitute the boundary of the area or the chaotic attractor).

In this section we consider two-dimensional endomorphisms $T : IR^2 \to IR^2$, of which the LC is constituted of two distinct branches, separates the phase plane in three open regions Z_1^1, Z_3 and Z_1^2. Z_3 is the place of points having three preimages of distinct first rank, Z_1^1 et Z_1^2 the one of points having only one antecedent. These applications are called of type $Z_1 - Z_3 - Z_1$.

We denote respectively, by D_0 and D immediate basin of attraction and total attraction basin of an attractor. The region not containing the attractor is called an island. It is interesting to know when basins are connected or nonconnected. The creation of holes inside the basin is considered as a bifurcation, that means a qualitative change in the system behavior. This can be explained on the basis of the critical curves properties and it is the same for the chaotic attractors and their bifurcations.

Proposition 3.1: *Consider an endomorphism T depending on a parameter λ. If the connected components number of $D \cap LC$ changes when λ crosses bifurcation value λ^*, then the basin D may undergo a qualitative change of one of the following types:*

(a) connected basin $\leftrightarrow$ non connected basin (when the number of connected components of $D_0 \cap LC$ changes).

(b) simply connected basin$\leftrightarrow$ multiply connected basin (when the number of connected components of $D_0 \cap LC$ changes).

(c) modification of the number of lakes in D, or new arborescent sequence of islands.

(d) destruction of a chaotic area.

Remark: Combined situations of (a) and (b) lead to a nonconnected total basin D and each of its connected components is multiply connected. That is, according to terminology of Mira [6], islands and lakes into islands.

4 Properties of $W^s(S)$ and $W^u(S)$

1– In the case of a diffeomorphism, the stable invariant manifold $W^s(S)$ of a saddle point S remains connected ; what is not true when one has an endomorphism. Indeed $W^s(S)$ constitutes in general the frontier of the basin of attraction of an associated attractor A. When $W^s(S)$ enters in contact with the critical line LC, $W^s(S)$ becomes nonconnected and each of its connected components approaches again itself to constitute the frontier of an island. It transforms the connected attraction basin in nonconnected basin. Due also to crossing of the invariant closed curve basin through LC, holes appear inside the basin, which becomes multiply connected. All these phenomena generate a very big sensitivity to initial conditions, due principally to multistability, interconnection between basins and fractalization of basins.

2– When unstable invariant manifold $W^u(S)$ enters in contact with the critical curve LC, a bifurcation making appear self-intersections of this manifold takes place. These self-intersections are responsible notably of the transformation of a closed invariant curve $\mathit{\Gamma}$ in a chaotic attractor. Indeed, a point of self-intersection of the curve is a point of non differentiability. By successive iterations of T, one will have an infinity of points of self-intersections, from points of non differentiability, what means that Γ becomes fractal. Its Liapounov dimension will be then superior to 1, what first generates an annular chaotic area, and a chaotic attractor.

5 Example of a Cubic Recurrence Having a Chaotic Attractor

Consider the cubic map T defined by

$$\begin{cases} x_{n+1} = x_n^3 + ax_n + b + y_n \\ y_{n+1} = cx_n + dy_n \end{cases}$$

where a, b, c, d are real parameters. This endomorphism is of type $Z_1 - Z_3 - Z_1$, whose critical curves LC_{-1}, LC'_{-1}, LC, $LC^{'}$ are given here by

$$\left(LC_{-1} : \ x = +\sqrt{\frac{c}{3d} - \frac{a+1}{3}}, \quad LC'_{-1} : \ x = -\sqrt{\frac{c}{3d} - \frac{a+1}{3}} \right)$$

and

$$\begin{pmatrix} LC: & y = d.x - d.x_0^3 - (a.d + d - c)x_0 - d.b \\ LC^{'}: & y = d.x + \ d.x_0^3 + (a.d + d - c)x_0 - d.b \end{pmatrix}$$

with

$$x_0 = \sqrt{\frac{c}{3d} - \frac{a+1}{3}}.$$

Despite the simplicity of the map, the concept is central to our interest and subject.

For fixed parameter values, we plot the attraction basin of an attractor. When there exist several attracting sets, it is possible to define a global basin, that means the set of initial conditions giving rise to bounded iterated sequences, independently of the fact that the converge to one attractor or another.

I) First let us fix parameters a, b, c at some real values and we vary the parameter d. For the values $a = -1$, $b = 1$, $c = -0.9$ and d varying in the decreasing sense, one has the following situations:

1. For $d = -0.35$ (Fig.1), the attractor is an attractive invariant closed curve (CFIs); which results from a Neimark - Hopf bifurcation. The connected component (the island) H_0 of basin of attraction has no contact with the branch $LC^{'}$; it is completely in the region Z_1.

2. For $d = -0.36$ (Fig.2), the island H_0 has a contact with $LC^{'}$ and enters in the region Z_3. This gives rise to a birth of an island H_{-1} and to an arborescent sequence of islands.

3. For $d = -0.47$ (Fig.3), the attractor remains CFIs. This one is more and more near the branch LC.

4. Pour $d = -0.48$ (Fig.4), The CFIs degenerates after a contact with the branch LC. The attractor becomes a set of points of dimension zero. On the Figure 5, one sees that the degeneration takes place when $W^u(S)$ touchs LC. This contact gives, after bifurcation, self-intersections of $W^u(S)$, which will transform the CFIs in a chaotic attractor, the size of which increases, until there is a contact between the curve and a part of its basin.

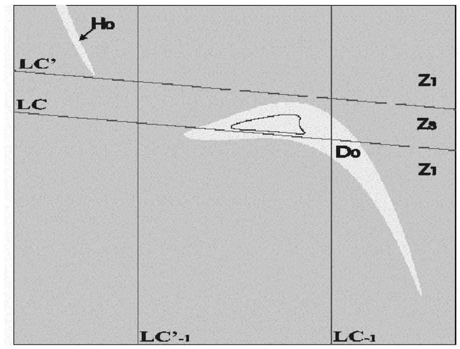

Fig. 1. Critical curves and Z_i areas in the state plane.

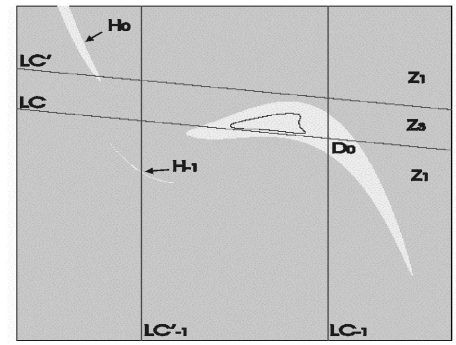

Fig. 2. Contact bifurcation of the island H_0 and apparitin of H_{-1}.

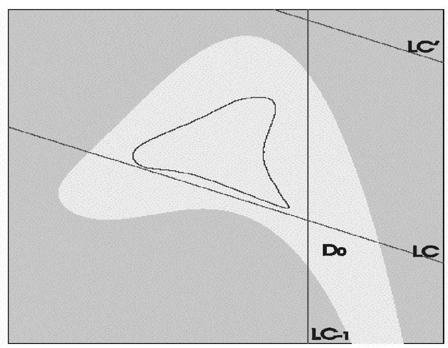

Fig. 3. Contact bifurcation of the attractor with the critical curve.

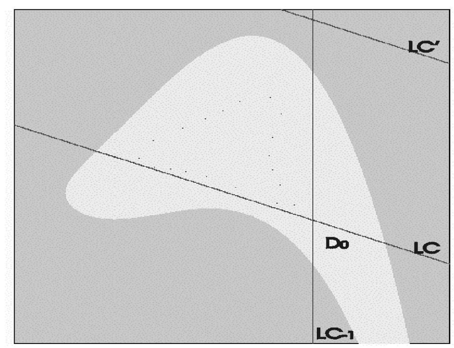

Fig. 4. Attractor of dimension zero.

II- In the following one will fix $c = -1.003$, to visualize better our figures.

5. For $d = -0.76$ (Fig.6), we can see the presence of a chaotic attractor, of which the dimension of Liapounov D_L is equal to 1.52. Its boundary doesn't have a contact with the boundary of its attraction basin. It results the

   Proposition 5.1 : *A cascade of bifurcations flip of cycles of order* $2^i.3$ *occurs.*

   d decreases, the period-3 orbit undergoes a period doubling bifurcation (see Fig.9) and gives up its stability and there is a sequence of period doubling bifurcations. Remarkably the ratio of the distances between these successive period doubling bifurcations again approaches. The Figure 9 gives the parameter value for which at least one fixed point is attractive (blue domain corresponding to the value 1). More generally,

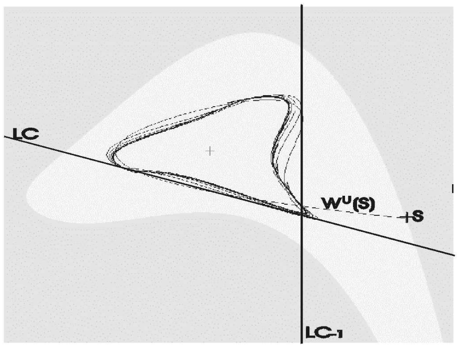

Fig. 5. The closed curve become chaotic.

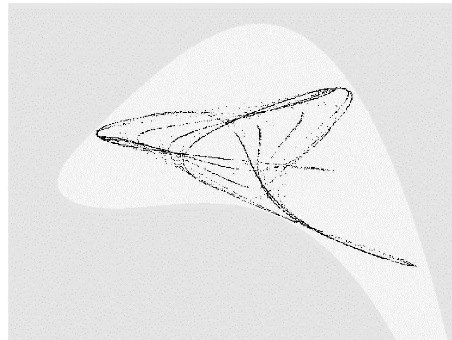
Fig. 6. Chaotic attractor.

the Figure 9 gives the regions of parameter (a, d) plane for which at least a periodic orbit of order k exists ($k = 1, 2, ..., 14$). The black regions ($k = 15$) corresponds to the existence of bounded iterated sequences. This figures is typical of maps with dominating cubic terms, analogous bifurcation diagrams have already been obtained for systems with non linearity given by hyperbolic tangency [3, 4].We can recognize on the diagram period doubling bifurcation, there exist several ways in which a dynamical system can become chaotic, of which the period-doubling route to chaos is the best known and identified here. The bifurcation structure is a cubic " box-within-a-box " type, as is well known infinitely many periodic are opened by fold bifurcations and are closed by homoclinic bifurcations by the intriguing " box-within-a-box " bifurcation structure. When chaotic motion appears as the result of a finite or infinite number of bifurcations, where some periodic regimes loss their stability, the final chaotic attractor has a close relation to stable and unstable manifolds. Geometrically, it is easy to imagine that at least one branch of unstable manifold will approach the attractor, while the stable manifold traced backwards, will outline the boundary of the basin of attraction.

6. For $d = -0.8$ (Fig. 7), The chaotic attractor has a contact with the boundary of its basin of attraction ; it is a contact bifurcation. These points of contact are also homoclinic points, since they are points of contact between $W^u(S)$ and $W^s(S)$; and then we are in presence of a homolinic bifurcation. $D_L = 1.72$.

7. For $d = -1.498$ (Fig.8), The chaotic attractor becomes larger, since $D_L = 2$ and below this value of d the attractor disappears.

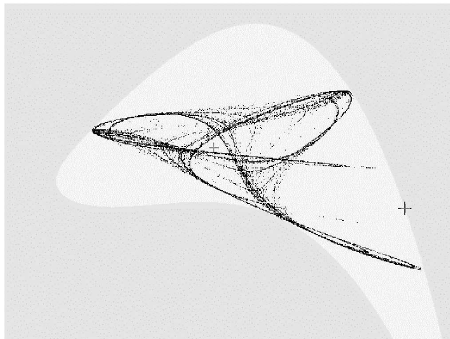

Fig. 7. Contact bifurcation of the chaotic attractor with its basin boundary.

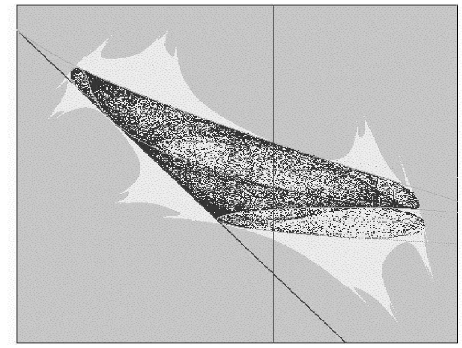

Fig. 8. Granulated attractor and tongues appear on the bassin boundary.

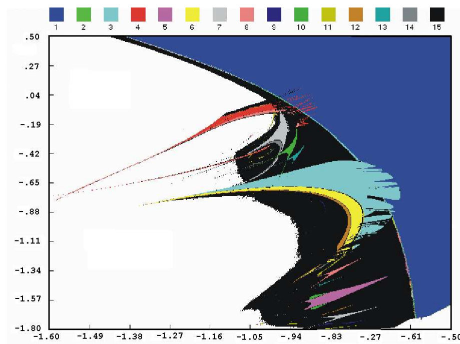

Fig. 9. Global view of the parameter plane (a, d) organisation.

The crossing of the bifurcation value $d = -1.5$ leads to a destruction of the chaotic attractor and the disappearing of its basin, complex homoclinic situations have before occured leading to a granulated basin corresponding to the presence of infinitely many sequences of homoclinic and heteroclinic points. Between these two values of d ($d = -1.498$, $d = -1.5$) the basin undergoes successive bifurcations creating more of holes. These complex phenomena arise in very small regions of parameter variations.

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