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Center Manifold in Continuous Time Systems and Computation

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Abstract: The objective in this paper is to give some results of bifurcation equations, is concerned with the bifurcation from an equilibrium point in the case when the linear approximation has eigenvalues with zero real parts. As we know, there is an intimate relationship between changes of stability and bifurcation. We formulate the main theorems that allow one to reduce dimension of a given system near a local bifurcation. We treat only continuous case.

Center manifold theory is a method which uses power series expansions in the neighborhood of an equilibrium point in order to reduce the dimension of a system of ordinary differential equation. We will discuss some aspects of the center manifold. In this paper we will be concerned with the question of how to reduce a system to its center manifold. The calculation of center manifolds involves the manipulation of truncated power series. Coefficients of the quadratic Taylor expansion representing the center manifold can be computed via a recursive procedure, each step of which involves solving a linear system of algebraic equations.

We present programs by Maple to accomplish such computations.

Keywords: Bifurcations, reduction, solutions, approximation.

1 Introduction

Consider differential equations of the form

$$\dot{v}= f\left(v\right) = Av+ \widetilde{f}\left(v\right) \tag{1.1}$$

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with $v \in R^n, f \in C^k(\mathbb{R}^n)\ (k \geq 1), f(0) = 0,\ A = Df(0) \in \pounds(\mathbb{R}^n)$, and hence $\widetilde{f}\ (0) = D\ \widetilde{f}\ (0) = 0$. For each $v \in \mathbb{R}^n$ we denote by $t \to \widetilde{v}\ (t, v)$ the unique maximal solution of (1.1) satisfying $v(0) = 0$.

The first step in any study of the flow of (1.1) near the singular point $v = 0$ is to linearize ; one consider the linear equation

$$\dot{v} = Av \tag{1.2}$$

The behavior of the solutions $\widetilde{x}_0\ (t, x) = \exp(At)\, x$ of (1.2) is completely determined by the spectral properties of A .

Since these spectral properties will play a central role in our further theory we introduce some notations. We denote by $\sigma = \sigma(A) \subset C$ the set of eigenvalues of A ; this spectrum is the disjoint union of the stable spectrum σ_s, the central spectrum σ_c, and the unstable spectrum σ_u, where

$$\begin{cases} \sigma_s = \{\lambda \in \sigma(A) \,/\, \operatorname{Re} \lambda < 0\} \\ \sigma_c = \{\lambda \in \sigma(A) \,/\, \operatorname{Re} \lambda = 0\} \\ \sigma_u = \{\lambda \in \sigma(A) \,/\, \operatorname{Re} \lambda > 0\} \end{cases}$$

Let X_s be the subspace of $\mathbb{R}^n$ spanned by the generalized eigenvectors of A corresponding to eigenvalues $\lambda \in \sigma_s$; in a similar way we define subspaces X_c and X_u. We have

$$\mathbb{R}^n = X_s \oplus X_u \oplus X_c \tag{1.3}$$

All bounded solutions, in particular equilibria and periodic solutions, belong to the center subspace X_c. Moreover, if $\sigma_u = \varnothing$ and hence X_u is trivial, then all solutions of (1.2) converge exponentially to a solution in the center subspace, and stability considerations can be restricted to the flow on X_c.

Behavior of the solutions of (1.2) has certainly a relevance for the system (1.1) ; in particular, there is resemblance between the local flow of (1.1) near $v = 0$ and the flow of (1.2) . In the case of a hyperbolic singular point, i.e. when $\sigma_c = \emptyset$k; the Hartman-Grobman theorem [3] states that the flows of (1.1) and (1.2) are topologically equivalent near $v = 0$. When the singular point is non-hyperbolic the qualitative behavior of (1.1) will also depend on the nonlinear terms of $f(v)$. We restrict our intention to an invariant subspace which contains all of the essential behavior of the system in the neighborhood of the equilibrium point.

We will interest on the center manifold $(W^c_{loc}(0))$ which is of primordial importance for the study of bifurcation phenomena (see [2]). This center manifold has properties similar to those of the center subspace X_c for the linear equation (1.2) : $W^c_{loc}(0)$ contains all solutions of (1.1) which stay for all $t \in \mathbb{R}$ in a sufficiently small neighborhood of the equilibrium $v = 0$, such as small equilibria, periodic solutions, heteroclinic or homoclinic solutions, etc. Also, if $\sigma_u = \emptyset$ then all solutions of (1.1) which stay near $v = 0$ as $t \to \infty$ will converge exponentially to some solution on $W^c_{loc}(0)$.

The center manifold approach consists to reduce the dimension of the problem in $\mathbb{R}^n$ to the dimension of the center manifold which is equal to the number of critical eigenvalues of A . The center manifold theorem in finite dimensions has been proved by Pliss [8] and by Kelley [5]. Modern proofs can be found in Carr [1] and Vanderbauwhede [11]. The existence of center manifolds for several classes of partial differential equations and delay differential equations has been established during the last two decades.

The main technical steps of such proofs are to show that the original system can be formulated as an abstract ordinary differential equation on an appropriate function space and to use variation of constants formula to prove that this equation defines a smooth dynamical system on this function space. Several authors have been attracted by the idea of the center manifold and have treated partial differential equations which exhibit Hopf bifurcation by reducing them to ordinary differential equations in the center manifold; we mention the works of Hassart [4] in the investigation of periodic solutions. Through our process to establish algorithms, which is the purpose of the present work, we review and use the development of analysis techniques for continuous-time case since works of Sijbrand [10].

The subject of bifurcation theory is the analysis of families of non trivial solutions which branch off the trivial solution of

$$\dot{x} = A_\lambda x + \tilde{f}_\lambda(x) \tag{1.4}$$

Where $v \in \mathbb{R}^n$, A_λ is a linear part which depends on the real parameter $\lambda \in \mathbb{R}^k$, $\tilde{f}_\lambda$ is a non linear function which is at least quadratic on x for x close to zero.

A family of stationary solutions may bifurcate off the family of trivial solutions ($x = 0$) of (1.4) at some values of the parameter λ_0 for which A_{λ_0} has eigenvalues with zero real parts.

Such cases can be handled by embedding the given system in a larger system which contains the parameter λ as an additional dependant variable. Naturally the results are only valid for small values of the parameter λ close of λ_0.

The equation (1.4) can be brought into the form (1.1) by adding to (1.4) with the equation $\dot{\lambda}= 0$; we identify

$$\begin{cases} v = (x, \lambda) \\ Av = (A_{\lambda_0} x, 0) \\ \widetilde{f}\ (v) = \left(\widetilde{f_\lambda}\ (x) + A_\lambda x - A_{\lambda_0} x, 0 \right) \end{cases} \tag{1.5}$$

where the dimensions of the kernels of A and A_{λ_0} differ by 1.

In bifurcation problems it is sufficient to study the flow on the center manifold, which gives a considerable reduction of the system.

Some properties of center manifold, particularly those related to the nonuniqueness, the limited differentiability and nonanalyticity are subtle, profitable when they are verified and not yet fully understood. Fine structures of flows and bifurcations near an equilibrium point are determined but with certain added complications, with techniques resemble that of stable and unstable manifolds near a saddle.

This paper aims the study of these subtle properties which deal with the problem of uniqueness and differentiability and to present the cut-off technique responsible of the selection of a unique center manifold .

This work is divided in three sections. In the second section, we regard these properties, we state the necessary transformations and modifications of the original equation leading to the center manifold, we give some conditions which guarantee the uniqueness and C^∞center manifold. In the third section, we will be concerned with the question of how to reduce a system to its center manifold . The calculation of center manifolds involves the manipulation of truncated power series.We give two methods and we present programs by MAPLE to accomplish such computations.

2 Properties of the center manifold

The center manifold properties associated with the basic questions of existence, uniqueness, analyticity and differentiability can be of great interest in bifurcation theory particularly in application for a complete description of the solutions close to the bifurcation point.

2.1 Existence

In this section we formulate more explicitly the assumptions concerning the differential equation and we write down a number of modifications of the dynamical system

$$\dot{v} = f(v) = Av + \tilde{f}(v) \qquad v \in \mathbb{R}^n, \quad f \in C^k \tag{2.1}$$

For proving the existence of center manifold.

Let the eigenvalues of the Jacobian matrix A be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Suppose the equilibrium is not hyperbolic, assume that:

$$Card(\sigma_s) = n_s, \quad Card(\sigma_u) = n_u, \quad Card(\sigma_c) = n_c \tag{2.2}$$

because X_s, X_u, X_c are $A-$invariant, i.e. $A(X_{s,u,c}) \subset X_{s,u,c}$.

Define

$$A_{s,u,c} = A/X_{s,u,c}$$

and system (2.1) can be written as

$$\begin{cases} \dot{x_c} = A_c x_c + \tilde{f_c}(x_c, x_s, x_u) & ; x_c \in X_c \\ \dot{x_s} = A_s x_s + \tilde{f_s}(x_c, x_s, x_u) & ; x_s \in X_s \\ \dot{x_u} = A_u x_u + \tilde{f_u}(x_c, x_s, x_u) & ; x_u \in X_u \end{cases} \tag{2.3}$$

Theorem 2.1

There is a locally defined smooth n_c-dimensional invariant manifold $W^c_{loc}(0)$ of (2.1) that is tangent to X_c at $v = 0$, defined as

$$W^c_{loc}(0) = \{(x_c, \varphi(x_c)) \qquad : x_c \in U\}$$

where U is an open neighborhood of 0 in X_c and $\varphi \in C^k(U \to X_s \oplus X_u)$. Moreover, if $f \in C^\infty$ then for all $k \geq 1$. There is U_k such that $\varphi \in C^k(U_k \to X_s \oplus X_u)$.

Definition 2.1

The manifold $W^c_{loc}(0)$ is called a C^k $(k \geqq 1)$ center manifold.

Proof of the Theorem 2.1

Let $\chi : [0, \infty] \to R$ be a C^∞ function with the additional properties

$$\begin{cases} 0 \leq \chi(v) \leq 1 & \text{for all } v \geq 0 \\ \chi(v) = 1 & \text{for all } v \in [0, 1] \\ \chi(v) = 0 & \text{for all } v \in [2, \infty[\end{cases} \tag{2.4}$$

Let ε be an arbitrary positive parameter. Consider the equations

$$\begin{cases} \dot{x_c} = A_c x_c + \widetilde{f_c}\,(x_c, x_s, x_u)\,\chi\left(\frac{||x_c||}{\varepsilon}\right) \\ \dot{x_s} = A_s x_s + \widetilde{f_s}\,(x_c, x_s, x_u)\,\chi\left(\frac{||x_c||}{\varepsilon}\right) \\ \dot{x_u} = A_u x_u + \widetilde{f_u}\,(x_c, x_s, x_u)\,\chi\left(\frac{||x_c||}{\varepsilon}\right) \end{cases} \tag{2.5}$$

For searching a manifold W_ε of (2.3) of the form.

$$W_\varepsilon = \{(x_c, h_{s,\varepsilon}(x_c), h_{u,\varepsilon}(x_c)) \qquad : x_c \in X_c\} \tag{2.6}$$

where $h_{s,\varepsilon}$,$h_{u,\varepsilon}$ are a C^k function from X_c to $X_{s,u}$. We shall derive a formula for $h_{s,\varepsilon}(x)\,, h_{u,\varepsilon}(x)$.

Let $x_c(t_0) \in X_c$ and suppose that the x_c-component of the solution of (2.3) remains inside X_c for $t_0 \leq \omega \leq t$.

Using the invariance property of the center manifold, we have

$$\begin{aligned} &h_{s,\varepsilon}(x_c(t)) = x_s(t) = e^{A_s(t-t_0)} h_{s,\varepsilon}(x_c(t_0)) \\ &+ \int_{t_0}^{t} e^{A_s(t-\omega)}\, \widetilde{f_s}\,(x_c(\omega), h_{s,\varepsilon}(x_c(\omega)), h_{u,\varepsilon}(x_c(\omega)))\,\chi\left(\frac{||x_c||}{\varepsilon}\right) d\omega \end{aligned} \tag{2.7}$$

By choosing $t = 0$ and taking the limit for $t_0 \to -\infty$ we obtain

$$h_{s,\varepsilon}(x_c(0)) = \int_{-\infty}^{0} e^{-A_s\omega}\, \widetilde{f_s}\,(x_c(\omega), h_{s,\varepsilon}(x_c(\omega)), h_{u,\varepsilon}(x_c(\omega)))\,\chi\left(\frac{||x_c||}{\varepsilon}\right) d\omega \tag{2.8}$$

Similarly, for h_u we obtain

$$h_{u,\varepsilon}(x_c(0)) = -\int_{0}^{+\infty} e^{-A_u\omega}\, \widetilde{f_u}\,(x_c(\omega), h_{s,\varepsilon}(x_c(\omega)), h_{u,\varepsilon}(x_c(\omega)))\,\chi\left(\frac{||x_c||}{\varepsilon}\right) d\omega \tag{2.9}$$

While $x_c(t)$ has to satisfy for $-\infty < t < \infty$:

$$\dot{x_c} = A_c x_c + \widetilde{f_c}\,(x_c, h_{s,\varepsilon}(x_c), h_{u,\varepsilon}(x_c))\,\chi\left(\frac{||x_c||}{\varepsilon}\right) \tag{2.10}$$

Taking the limit $t_0 \to \pm\infty$ makes sense only if the quantity $h_{s,\varepsilon}(x_c(t_0))$, $h_{u,\varepsilon}(x_c(t_0))$ remains bounded when $t_0 \to \pm\infty$, Then we shall define $h_{s,\varepsilon}$,$h_{u,\varepsilon}$ bounded on all X_c.

If W_ε is a center manifold of (2.3) of the kind (2.4), then the restriction of W_ε to $U \cap D_\varepsilon$ (where D_ε is the ball of radius ε around 0 in X_c) is a center manifold of (2.1).

We perform a scaling

$$\begin{array}{lll} x_c = \varepsilon x & , x_s = \varepsilon y & , x_u = \varepsilon z \\ h_{s,\varepsilon} = \varepsilon \varphi_s & , h_{u,\varepsilon} = \varepsilon \varphi_u & \\ \tilde{f}_{c,s,u}\left(x_s, x_c, x_u\right) \chi\left(\frac{\|x_c\|}{\varepsilon}\right) = \varepsilon g_{c,s,u}\left(x, y, z; \varepsilon\right) & & \end{array}$$

The modified and scaled version of (2.6) (2.7) , (2.8) becomes

$$\left\{ \begin{array}{l} \varphi_s\left(\xi, \varepsilon\right) = \int\limits_{-\infty}^{0} e^{-A_s \omega} g_s\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varphi\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varepsilon\right); \varepsilon\right) d\omega \\ \varphi_u\left(\xi, \varepsilon\right) = -\int\limits_{0}^{\infty} e^{-A_u \omega} g_u\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varphi\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varepsilon\right); \varepsilon\right) d\omega \end{array} \right. \tag{2.11}$$

where $x\left(t; \xi, \varphi, \varepsilon\right)$ is the solution of

$$\dot{x} = A_c x + g_c\left(x, \varphi\left(x, \varepsilon\right), \varepsilon\right) \qquad x\left(0\right) = \xi \tag{2.12}$$

In the argument of $g_{c,s,u}$ we have abbreviated the pair φ_s, φ_u by φ.

The solution of (2.11) , (2.12) defines an invariant manifold. The tangency property is obvious. By the properties of the operator T_ε from [12] defined as follows

$$\begin{aligned} \left(T_\varepsilon \varphi\right)\left(\xi\right) = & \int\limits_{-\infty}^{0} \exp\left(-A_s \omega\right) g_s\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varphi\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varepsilon\right); \varepsilon\right) d\omega \\ & - \int\limits_{0}^{\infty} \exp\left(-A_u \omega\right) g_u\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varphi\left(x\left(\omega; \xi, \varphi, \varepsilon\right); \varepsilon\right); \varepsilon\right) d\omega \end{aligned}$$

we implies that $S \in C^k$ ■

2.2 Uniqueness

In general (2.1) does not have a unique center manifold. A simple example suffices to prove this, for example the trivial and well known system [5]

$$\left\{ \begin{array}{l} \dot{x} = x^2 \\ \dot{y} = -y \end{array} \right.$$

Let

$$h\left(x, c\right) = \left\{ \begin{array}{lll} c \exp\left(\frac{1}{x}\right) & \text{for} & x < 0 \\ 0 & \text{for} & x \geq 0 \end{array} \right.$$

Clearly $h\left(0, c\right) = h'\left(0, c\right) = 0$, so that $W\left(c\right) = \left\{\left(x, y\right) / y = h\left(x, c\right), x \text{ arbitrary}\right\}$ is a center manifold for each real constant c.

Theorem 2.2

Consider the equation

$$\begin{cases} \dot{x_c} = A_c x_c + \overset{\sim}{f_c}(x_c, x_s) \\ \dot{x_s} = A_s x_s + \overset{\sim}{f_c}(x_c, x_s) \end{cases} \tag{2.13}$$

Suppose that φ_1, φ_2 are C^k functions from a neighborhood U of 0 in X_c to X_s such that

$$W_1 = \{(x_c, \varphi_1(x_c)) : \quad x_c \in U\}$$
$$W_2 = \{(x_c, \varphi_2(x_c)) : \quad x_c \in U\}$$

are center manifolds of (2.13) . Let $x_c(t)$ denotes a solution of

$$\dot{x_c} = A_c x_c + \widetilde{f_c}(x_c, \varphi_1(x_c)) \tag{2.14}$$

Then there is a neighborhood $U' \subset U$ of 0 such that

$$\{x_c(t) \quad : \quad -\infty < t \leq 0\} \subset U' \tag{2.15}$$

implies

$$\varphi_1(x_c(t)) = \varphi_2(x_c(t)) \quad \text{pour } -\infty < t \leq 0 \tag{2.16}$$

Proof

According to the proof of theorem (2.1) there is a modification of (2.13) and a neighborhood $U' \subset U$ such that W_2/U' is a submanifold of the globally unique center manifold of

$$\begin{cases} \dot{x_c} = A_c x_c + \widetilde{g_c}(x_c, x_s) \\ \dot{x_s} = A_s x_s + \widetilde{g_s}(x_c, x_s) \end{cases} \tag{2.17}$$

Consider the iteration defined as

$$\varphi^{(n+1)}(\xi) = \int_{-\infty}^{0} \exp(-A_s\omega)\, \widetilde{g_s}\left(x_c\left(\omega; \varphi^{(n)}\right); \varphi^{(n)}\left(x_c\left(\omega; \varphi^{(n)}\right)\right)\right) d\omega \tag{2.18}$$

Such that $x_c(t; \varphi)$ is the solution of

$$\begin{cases} \dot{x_c} = A_c x_c + \widetilde{g_c}\left(x_c, \varphi^{(n)}(x_c)\right) \\ x_c(0) = \xi \end{cases} \tag{2.19}$$

¿From the contraction property of the operator T_ε this iteration converges to $\varphi_2(\xi)$ if $\left\|\varphi^{(0)}\right\|$ is small enough and $\xi \in U'$. Now let us start this iteration with $\varphi^{(0)} = \varphi_1$. By the assumption (2.15) made on the solution of (2.14) we have, for $\xi = x_c(0;\varphi_1)$

$$\begin{aligned}\varphi^{(1)}(\xi) &= \int_{-\infty}^{0} \exp(-A_s\omega)\, \tilde{f}_s\left(x_c(\omega;\varphi_1);\varphi_1(x_c(\omega;\varphi_1))\right) d\omega \\ &= \int_{-\infty}^{0} \exp(-A_s\omega)\left(\frac{d}{d\omega} - A_s\right)\varphi_1(x_c(\omega;\varphi_1))\, d\omega \\ &= \varphi_1(\xi)\end{aligned}$$

Hence, all iterands $\varphi^{(n)}(\xi)$ are equal to $\varphi_1(\xi)$ which implies that $\varphi_2(\xi) = \lim_{n\to\infty} \varphi^{(n)}(\xi) = \varphi_1(\xi)$. ■

Remark 2.1

This theorem gives the condition for the uniqueness in few words the center manifold is unique there where the trajectories stay close to the singular point for $t \to -\infty$.

Theorem (2.2)'

i) If the linear operator A in (2.1) has $\sigma_s = \emptyset$, then the statement (2.16) holds for $0 \le t \le \infty$ if

$$\{x_c(t) \quad : \quad 0 \le t < \infty\} \subset U' \tag{2.15'}$$

ii) If A has eigenvalues with negative and positive real parts, then the statement (2.16) holds for $-\infty \le t \le \infty$ if

$$\{x_c(t) \quad : \quad -\infty < t < \infty\} \subset U' \tag{2.15''}$$

Proof:

The method of proof is the same technique as before ■

Corollary 2.1

Consider the system (2.1) where $\tilde{f} \in C^k\ (k \ge 1)$, Let W_1, W_2 two C^{k+1} center manifolds of (2.1) defined as:

$$\begin{aligned}W_1 &= \{(x_c, \varphi_1(x_c)) : \quad x_c \in U\} \\ W_2 &= \{(x_c, \varphi_2(x_c)) : \quad x_c \in U\}\end{aligned}$$

Then

$$D^j\varphi_1(0) = D^j\varphi_2(0) \quad \text{pour } 1 < j \le k$$

Remark 2.2:

According to the corollary Taylor series of two different center manifolds of dynamical system are the same.

Differentiability

Now for the differentiability of the center manifold (see [10, 11]) we shall investigate a class of problems where the existence of a C^∞ center manifold is guaranteed.

Theorem 2.3

Consider the system (2.13) where $\tilde{f}_{s,c} \in C^\infty$, suppose that
1) $\dim X_c = 1, (A_c = 0)$ and 0 is locally an isolated stationary point.
or
2) $\dim X_c = 2$ and $A_c = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and 0 is locally an isolated periodic solution.
Let W denotes a C^1 center manifold of (2.13) defined on a neighborhood U_0 of the singular point "0" .
Then there is a neighborhood $U' \subset U_0$ such that $\varphi \in C^\infty(U' \to X_s)$.

Proof

1) suppose that $\dim X_c = 1$. Because 0 is an isolated stationary point there is a neighborhood $U_1 \subset U_0$ such that the function F defined by $F(x_c) = \tilde{f}_c(x_c, \varphi(x_c))$ has no zeros in U_1 except at the origin.

Let the interval $[0, a_3) \subset U_1$. If $F > 0$ on $(0, a_3)$ then the part of W above $[0, a_3)$ is unique, and for φ there are open neighborhoods $U_k \subset [0, a_3), (k \ge 1)$ such that $\varphi \in C^k(U_k \to X_s)$. Let $x_{c0} \in [0, a_3)$ be arbitrary.

If $(x_c(t;\xi), x_s(t;\xi))$ denotes the solution of the initial value $x_c(0;\xi) = \xi$, $x_s(0;\xi) = \varphi(\xi)$. then there is a point $x_{ck} \in U_k$ and a number $t_k > 0$ such that $x_c(t_k, x_k) = x_{c0}$. Because φ is a C^k in an open neighborhood $E_k \subset U_k$ of x_k, the set

$$\{(x_c(t_k;\xi), x_s(t_k;\xi)) : \xi \in E_k\}$$

is a C^k manifold. But this set represents the center manifold in a neighborhood of $(x_{c0}, \varphi(x_{c0}))$. So the center manifold is C^k on all $[0, a_3)$, then $\varphi \in C^\infty([0, a_3))$

If $F < 0$ on $(0, a_3)$, then we modify the system (2.13) by a C^∞ function Φ such that $0 < a_1 < a_2 < a_3$ with

$$\begin{cases} \Phi(x) = 1 & \text{for} \quad 0 \le x \le a_1 \\ \Phi(x) > 0 & \text{for} \quad a_1 \le x \le a_2 \\ \Phi'(a_2) < 0 & \Phi(a_2) = 0 \\ \Phi(x) = 0 & \text{for} \quad x \ge a_3 \end{cases} \tag{2.20}$$

The modified system

$$\begin{cases} \dot{x_c} = \widetilde{f}_c(x_c, x_s).\Phi(|x_c|) \\ \dot{x_s} = A_s x_s + \widetilde{f}_s(x_c, x_s).\Phi(|x_c|) \end{cases} \tag{2.21}$$

will have a unique global C^1 center manifold $W^* = \{(x_c, \varphi^*(x_c)) : x_c \in X_c\}$ and 0 is an attractor for $t \to \infty$ in $W^*/\,[0, a_3)$. Let x_{c1} be the x_c-coordinate of the first singular point of (2.21) which lies to the right of 0 in W^* : $0 < x_{c1} \le a_2$ (the point x_{c1} not necessarily coincides with a_2). Then the restriction of φ to $[0, x_{c1}]$ is either the unstable manifold of (2.21) at x_{c1}or it is the center manifold at x_{c1} with the additional property that for $t \to -\infty$, x_1 is an attractor. Then $\varphi \in C^\infty([0, x_{c1}] \to X_s)$. On the interval $(-a_3, 0]$ the argument are similar.

For the second case the proof is analogous to the foregoing proof ■

Corollary 2.2 (due to F.Dumortier)

Let φ and F be as in proof of theorem above suppose $\varphi \in C^k(U_k \to X_s)$ and $F(x) = x^p\, \widetilde{F}(x), p \le k, \widetilde{F}(0) \ne 0$. Then for a neighborhood U_2 of 0 with $U_2 \subset U_0$ we have $\varphi \in C^\infty(U_2 \to X_s)$.

Proposition 2.1

Suppose that $\dim X_c = 1$, Let $\widetilde{f}_{c,s} \in C^\infty$ and $\varphi(x)$ represent a C^k-center manifold of (2.1) $(k \ge 1)$.

If the $p$ieme derivative $(p \le k)$ of $\widetilde{f}_c(x_c, \varphi(x_c))$ does not vanish at $x_c = 0$, then $\varphi \in C^\infty$.

2.3 Analyticity

Because the analytic center manifold play a great role in the study of bifurcation theory, we have investigated these results.

Proposition 2.2

Suppose that $\dim X_c = 1$, if all the derivative of $\widetilde{f}_c\left(x_c, \varphi\left(x_c\right)\right)$ vanish at $x_c = 0$, then φ is analytic.

Proposition 2.3

Unicity does not imply the Analyticity.

3 Computation of the center manifolds

As mentioned in the introduction the center manifold is of primordial importance for the study of bifurcation phenomena. This approach consists to reduce the dimension of the problem in $\mathbb{R}^n$ to the dimension of the center manifold n_c.

Since bifurcation results essentially depend on some higher order terms in the vector field (see [5, 6, 11, 12] for details), it is important that the center manifold is sufficiently smooth such that these higher order terms make sense for the reduced system. Moreover, one should be able to calculate these terms which means that one should be able to approximate the center manifold to sufficiently high order.

There are useful methods for center manifold computation which avoid the transformation of the system into its eigenbasis. The first method is known and uses power series expansions in the neighborhood of an equilibrium point in order to reduce the dimension, it involves restricting attention to an invariant subspace (the center manifold) which contains all of the essential behavior of the system in the neighborhood of an equilibrium point. The second method projects the system into the critical eigenspace and its complement. This projection technique, which avoids putting the linear part in normal form, was originally developed to study bifurcation in some partial differential equations using the Liapunov-Schmidt reduction [1].

Our representation is based on the book by Hassart et al. [4] where the Hopf bifurcation is treated in detail. Computational formulas for the discrete time flip case are given by Kuznetsov [6].

3.1 Method of polynomial approximation

Consider the dynamical system (2.1) such that $f \in C^k(\mathbb{R}^n) \quad (k \geq 1)$, and 0 is an isolated singular point and not hyperbolic then we can write (2.1) in the form (2.3) where $A_c\,(A_s, A_u)$ is an matrix with all its eigenvalues in $\sigma_c\,(\sigma_s, \sigma_u)$ respectively.

According to the existence theorem of the center manifold, there is $\varphi :$ $U \subset X_c \to X_s \oplus X_u$ such that $\varphi(0) = D\varphi(0) = 0$ and

$$\varphi(x_c) = (h_s(x_c), h_u(x_c)) = (x_s, x_u) \tag{3.1}$$

(3.1) represents the center manifold of the system (2.1) at 0 where U is an open neighborhood of 0.

We substitute $x_s = h_s(x_c)$ and $x_u = h_u(x_c)$ in the second and the third equation of (2.1) :

$$\begin{aligned} \frac{\partial h_s}{\partial x_c}\,\dot{x_c} &= A_s h_s(x_c) + \tilde{f}_s(x_c, h_s(x_c), h_u(x_c)). \\ \frac{\partial h_u}{\partial x_c}\,\dot{x_c} &= A_u h_u(x_c) + \tilde{f}_u(x_c, h_s(x_c), h_u(x_c)). \end{aligned} \tag{3.2}$$

In the center manifold we have

$$\dot{x_c} = A_c x_c + \tilde{f}_c(x_c, h_s(x_c), h_u(x_c)) \tag{3.3}$$

Therefore, we obtain a partial differential equation for φ

$$N(h_s(x_c)) = \frac{\partial h_s}{\partial x_c}\left(A_c x_c + \tilde{f}_c(x_c, h_s, h_u)\right) - A_s h_s - \tilde{f}_s(x_c, h_s, h_u) = 0 \tag{3.4}$$

$$N(h_u(x_c)) = \frac{\partial h_u}{\partial x_c}\left(A_c x_c + \tilde{f}_c(x_c, h_s, h_u)\right) - A_u h_u - \tilde{f}_u(x_c, h_s, h_u) = 0 \tag{3.5}$$

With $h_{s,u}(0) = 0$ and the tangency conditions $\dfrac{\partial h_{s,u}}{\partial x_c}(0) = 0.$

Generally, we can't solve this equation.

Theorem 3.1 (Carr [1])

If a function $\phi(x_c)$, with $\phi(0) = D\ \phi(0) = 0$, can be found such that $N(\phi(x_c)) = 0(|x_c|^p)$ for some $p > 1$ as $|x_c| \to 0$ then it follows that $\varphi(x_c) = \phi(x_c) + 0(|x_c|^p)$ as $|x_c| \to 0$.

Thus we can approximate $\varphi(x_c)$ as closely as we wish by seeking series solutions of $(3.4 - 3.5)$. However, such Taylor series expansions do not always exist, since W^c may not be analytic consequently we have this result.

Proposition 3.1

Center manifold is analytic if and only if its Taylor series expansions converge. In the bifurcation case the matrix A_c, A_s, A_u and functions $\widetilde{f}_c, \widetilde{f}_s, \widetilde{f}_u$ depend upon a parameter $\lambda \in \mathbb{R}^k$, and write the extended system as:

$$\begin{cases} \dot{\lambda} = 0 \\ \dot{x_c} = A_{c,\lambda} x_c + \widetilde{f}_{c,\lambda}(x_c, x_s, x_u) \\ \dot{x_s} = A_{s,\lambda} x_s + \widetilde{f}_{s,\lambda}(x_c, x_s, x_u) \\ \dot{x_u} = A_{u,\lambda} x_u + \widetilde{f}_{u,\lambda}(x_c, x_s, x_u) \end{cases} \quad (x_c, x_s, x_u, \lambda) \in X_c \times X_s \times X_u \times \mathbb{R}^k \quad (3.6)$$

At $(x_c, x_s, x_u, \lambda) = (0,0,0,0)$, (3.6) has an $n_c + k-$dimensional center manifold tangent to (x_c, λ) space, which may be approximated as the power series in x_c and λ .

3.2 Projection method

There is a useful method for center manifold computation which avoids the transformation of the system into its eigenbasis (to the form (2.1)).

Instead, only eigenvectors corresponding to the critical eigenvalues of A and its transpose $A^T$are used to project the system into the critical eigenspace X_c and its complement $(X_c)^c$. Let now presenting this method for the system (2.1) for the case where A has a simple zero eigenvalue $\lambda = 0$, and the corresponding critical eigenspace X_c is one-dimensional and spanned by an eigenvector $q \in \mathbb{R}^n$ such that $Aq = 0$. Let $p \in \mathbb{R}^n$ be the adjoint eigenvector, that is $A^T p = 0$, where A^T is the transposed matrix. That is possible and convenient to normalize p with respect to q : $\langle p, q \rangle = 1$, where $\langle ., . \rangle$ is the standard scalar product in $\mathbb{R}^n$.

Lemma 3.1(Fredholm Alternative)

Let X_h the hyperbolic subspace where $\dim X_h = (n-1)$. Then $y \in X_h$ if and only if $\langle p, y \rangle = 0$.

Using this lemma, we can decompose any vector $v \in \mathbb{R}^n$ as

$$v = uq + y$$

where $uq \in X_c$ et $y \in X_h$. If p and q are normalized as above one gets explicit expressions for u and y

$$\begin{cases} u = \langle p, v \rangle \\ y = v - \langle p, v \rangle q \end{cases} \tag{3.7}$$

Thus we can define two operators:

$$\begin{aligned} p_c v &= \langle p, v \rangle q \\ p_h v &= v - \langle p, v \rangle q \end{aligned} \tag{3.8}$$

These two operators are the projections onto X_c et X_h respectively . The scalar u and the vector y can be considered as new coordinates on $\mathbb{R}^n$. Hence the system (2.1) can be written as :

$$\begin{cases} \dot{u} = \left\langle p, \widetilde{f}\,(uq + y) \right\rangle \\ \dot{y} = Ay + \widetilde{f}\,(uq + y) - \left\langle p, \widetilde{f}\,(uq + y) \right\rangle q \end{cases} \tag{3.9}$$

Using Taylor expansions, we can write this last system in the form

$$\begin{cases} \dot{u} = \frac{1}{2}\alpha u^2 + u \langle b, y \rangle + \frac{1}{6}\mu u^3 + ... \\ \dot{y} = Ay + \frac{1}{2}a u^2 + ... \end{cases} \tag{3.10}$$

where $u \in \mathbb{R}$, $y \in \mathbb{R}^n$, $\alpha, \mu \in \mathbb{R}$, $a, b \in \mathbb{R}^n$ and $\langle b, y \rangle = \sum_{i=1}^{n} b_i y_i$ is the standard scalar product in $\mathbb{R}^n$. For α, μ, a, b we have the following expressions

$$\alpha = \frac{\partial^2}{\partial u^2} \left\langle p, \widetilde{f}\,(uq) \right\rangle |_{u=0} \tag{3.11}$$

$$\mu = \frac{\partial^3}{\partial u^3} \left\langle p, \widetilde{f}\,(uq) \right\rangle |_{u=0} \tag{3.12}$$

$$a = \frac{\partial^2}{\partial u^2} \widetilde{f}\,(uq)\,|_{u=0} - \alpha q \tag{3.13}$$

and the components of the vector b are given by:

$$b_i = \frac{\partial^2}{\partial y_i \partial u} \left\langle p, \widetilde{f}\,(uq + y) \right\rangle |_{u=0, y=0} \qquad i = 1, .., n \tag{3.14}$$

The center manifold has the representation

$$y = \varphi(u) = \frac{1}{2}\omega_2 u^2 + 0\left(u^3\right)$$

where $\omega_2 \in X_h \subset \mathbb{R}^n$ which means: $\langle p, \omega_2 \rangle = 0$, the vector $\omega_2$satisfies an equation in $\mathbb{R}^n$ that formally resemble at

$$A\omega_2 + a = 0 \tag{3.15}$$

The restriction of (3.11) to the center manifold takes the form

$$\dot{u} = \frac{1}{2}\alpha u^2 + \frac{1}{6}\left(\mu + 3\langle b, \omega_2\rangle\right) u^3 + 0\left(u^4\right) \tag{3.16}$$

Remark 3.1

The choice of normalization for q is irrelevant

3.3 Algorithms

We present two programs (see [7] for details) for calculating center manifold, of degree l,for systems of form (2.3) in the case $n_c = 1$ and $n_c = 2$. In addition to this various methods are displayed in [9] for MACSYMA system.

```
progra1 := proc(n, l)
local i, j, eq, temp, p, q;
print(eigenvalue − is − zero)
/*SET UP DIF. EQ.*/
for i from 1 to n
do eq(i) := diff(x[i](t), t) = (f[i] x(t));
print(eq(i));
od;
/*FORM POWER SERIES*/
for p from 2 to n
do
temp(p) := 0;
for i from 2 to l
do
temp(p) := temp(p) + a[p, i] * x[1](t)^i;
od;
od;
for p from 2 to n
do
/*SUBSTITUTE  POWER SERIES*/
for j from 2 to n
```

```
do
subs(x[j](t) = temp(j), eq(p));
subs(lhs(eq(1) == rhs(eq(1)), %);
for i from 2 to n
do
subs(x[i](t) = temp(i), %)
eq(p) := %;
od;
od;
od;
/*COLLECT TERMS*/
for p from 2 to n
do
var(p) := lhs(eq(p)) - rhs(eq(p));
var(p) := %;
collect(var(p), x[1](t));
var(p) := %;
od;
/*CALCULATION OF THE COEFFICIENTS*/
for p from 2 to n
do
for i from 2 to l
do
sol(p, i) := coeff(var(p), x[1](t), i) = 0;
od;
od;
for i from 2 to l
do
for p from 2 to n
do
solve(sol(p, i), a[p, i]); cao(p, i) := %;
if  i < l then
for j  from i + 1  to  l
do
for q from 2 to n
do
subs(a[p, i] = cao(p, i), sol(q, j));
sol(q, j) := %;
od;
od;
fi;
subs(a[p, i] = cao(p, i), temp(p));
temp(p) := %;
od;
od;
/*SOLVE FOR C. M.*/
```

```
print(center - manifold - is);
for p from 2 to n
do
print(x[p](t) = temp(p));
od;
for p from 2 to n
do
subs(x[p](t) = temp(p), eq(1));
od;
collect(eq(1), x[1](t));
eq(1) := %;
/*GET FLOW ON C.M.*/
print(flow - on - the - center - manifold - is);
print(eq(1));
end:
```

We consider now $n_c = 2$; the listing for the program in this case is

```
progra2 := proc(n, l)
local i, j, p, k1, k2, eq, temp, m, r, q;
print(eigenvalues - have - zero - real -parts);
/*SET UP D.E.*/
for i from 1 to n
do
eq(i) := diff(x[i](t), t) = (f[i] x(t));
print(eq(i));
od;
/*FORM POWER SERIES*/
for p from 3 to n
do
temp(p) := 0;
for m from 2 to l
do
for k1 from 0 to l
do
for k2 from 0 to l
do
if k1 + k2 = m then
temp(p) := temp(p) + a[p, k1, k2] * x[1](t)^k1 * x[2](t)^k2;
fi;
od;
od;
od;
od;
/*SUBSTITUTE POWER SERIES*/
for p from 3 to n
do
for j from 3 to l
```

```
do
subs(x[j](t) = temp(j), eq(p));
subs(lhs(eq(1) = rhs(eq(1)), %);
subs(lhs(eq(2) = rhs(eq(2)), %);
for i from 3 to n
do
subs(x[i](t) = temp(i), %)
eq(p) := %;
od;
od;
od;
/*COLLECT TERMS*/
for p from 3 to n
do
var(p) := lhs(eq(p)) - rhs(eq(p));
var(p) := %;
collect(var(p), x[1](t), x[2](t), distributed);
var(p) := %;
od;
/*CALCULATION OF COEFFICIENTS*/
for p from 3 to n
do
for m from 2 to l
do
for k1 from 0 to l
do
sar(p, k1) := coeff(var(p), x[1](t), k1);
for k2 from 0 to l
do
if k1 + k2 = m then
sol(p, k1, k2) := coeff(sar(p, k1), x[2](t), k2) = 0;
fi;
od;
od;
od;
od;
for m from 2 to l
do
for k1 from 0 to l
do
for k2 from 0 to l
do
if k1 + k2 = m then
for p from 3 to n
do
solve(sol(p, k1, k2), a[p, k1, k2]); cao(p, k1, k2) := %;
```

```
for r from 2 to l
do
for i from 0 to l
do
for j  from 0 to l
do
if  i + j = r  then
for q from 3 to n
do
subs(a [p, k1, k2] = cao(p, k1, k2), sol(q, i, j));
sol(q, i, j) := %;
od;
fi;
od;
od;
od;
subs(a [p, k1, k2] = cao(p, k1, k2), temp(p));
temp(p) := %;
od;
fi;
od;
od;
od;
/*SOLVE FOR C.M.*/
print(center − manifold − is);
for  p  from 3  to  n
do
print(x [p] (t) = temp(p));
od;
for  p  from  3   to  n
do
subs(x [p] (t) = temp(p), eq(1));  eq(1) = %;
od;
collect(%, x [1] (t), x [2] (t), distributed);
eq(1) := %;
for  p  from  3   to  n
do
subs(x [p] (t) = temp(p), eq(2));  eq(2) = %;
od;
collect(%, x [1] (t), x [2] (t), distributed);
eq(2) = %;
/*GET FLOW ON C.M.*/
print(flow − on − the − center − manifold − is);
print(eq(1));
print(eq(2));
end :
```

4 Conclusion

Coefficients of the Taylor expansion representing the center manifold are computed via a recursive method, each step of which involves solving a linear system of algebraic equations.For the latter method it suffices to know the eigenspace and its complement, the calculation is less complicated.

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