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3D Electrostatic Field Solution with Dual Mesh

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Abstract: The paper presents an application of the "Finite formulation of electromagnetic field" to the analysis of electrostatic field. This new formulation proposes a direct expression of Maxwell laws in terms of global variables (fluxes, circulations, etc.) and exploits duality present in these equations. In this way, it is well suited for the numerical solution of electromagnetic field and makes this process similar to the study of an electrical network. A field solver is thus implemented using tetrahedral elements and several comparisons versus analytical solutions are presented and discussed.

Keywords: Finite formulation of electromagnetic field, duality, electrostatic field.

1 Introduction

The numerical solution of electromagnetic field has been a topic of research for several decades and many numerical receipts are now well assessed and used in the industrial process of design. Notwithstanding the important achievements reached with these well established numerical methods, there is still room for research on the very basical aspects of analysis.

Well established numerical techniques are theoretically based on a differential formulation of electromagnetic field. Starting from this formulation, a discretization is performed on the problem domain and an approximation

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of the solution is looked for, for instance by means of function series expansions like in the finite element method.

A new approach has been proposed recently by Tonti [1] that does not make use of the differential formulation of electromagnetic field. Instead of working with electromagnetic field quantities, this formulation takes into account their integral forms, like fluxes and circulations. This new view of the electromagnetic theory is very well suited for the numerical analysis when it is projected on a proper space and time discretization. This discretization exploits the duality present in the Maxwell equations and thus allows to obtain a very efficient and accurate solution of the problem.

This is not, however, a complete innovative solution, numerical solutions of electromagnetic field based on similar approaches have been previously presented [2,3] and are extensively used. As a matter of fact however, these methods rely heavily on a cartesian subdivision of the space (structured mesh) which easily leads to orthogonal dual meshes.

The use of structured grids, even if very simple and efficient on the theoretical point of view, can become cumbersome when it is applied to real world problems where complex boundaries must be followed. The finite formulation proposes some hints to overcome this problem and some of them have been already implemented successfully [4]. This topic is nowadays of interest as it is testified by the papers presented at the recent conference COMPUMAG [5, 6, 7]. This paper presents a new contribution to this line of development, proposing a dual finite solution on unstructured grids applied to the study of electrostatic configurations.

The theoretical basis of finite formulation and of this particular implementation are given and results obtained in some configurations of interest are presented.

2 Finite Formulation of Electromagnetic Field

The finite formulation of electromagnetic field proposed by Tonti [1] writes the equations of electromagnetic fields directly in a finite or discrete form, without using the differential formalism. The variables used in this formulation are called global variables and are for instance voltages, fluxes, charges etc. These quantities can be effectively measured and can be seen also as integral quantities of the electric and magnetic fields. Without going through the properties of the global variables, it is evident that they can be easily associated to space elements, (voltages to lines, fluxes to surfaces etc.) and they can offer a suitable way for a numerical solution of the electromagnetic field problem when the domain of the problem is subdivided in a complex of cells characterized by simple geometrical structure.

In this paper only the formulation of the electrostatic problem will be considered, so that a subset of the finite formulation is taken into account.

2.1 Global variables

An electrostatic problem is considered, this hypothesis allows to eliminate the time dependence of the field variables and to introduce an electric scalar potential. Without going into details of the definition of the variables, which that can be found in [1], the variables needed to set-up the problem are:

- electric flux Ψ on a surface, this variable can be seen as the flux of the electric induction, $\Psi = \int_S DdS$;
- electro-motive force ξ on a line, also in this case $\xi = \int_l Edl$, where E is the electric field vector; and
- electric scalar potential u defined on points.

These variables are related to some spatial entities: flux are related to surfaces, e.m.f. to lines and potential to points.

2.2 Space discretization

The algorithm for the electromagnetic field solution requires that the field variables would be related to the space elements used to describe the domain of the problem. The main concept underneath the numerical implementation of the finite formulation is associated to the definition of two complexes of cells. These two sets of elements are intertwined so that each entity of one complex is univocally linked to one of the other. This link is usually referred to as duality. Considering a 3 dimensional space, four entities can be defined: points P, lines L, surfaces S and volumes V. Other elements can be considered also in the time domain but are not taken into account here because of the stationary hypothesis.

The definition of the elements starts from a primal cell complex whose entities are named as P, L, S and V, each of these quantities is characterized by a spatial orientation. Once the primal system has been defined, the dual one can be obtained considering that each p-dimensional entity of the primal system is linked to a n-p dimensional one of the dual one, where n is the number of dimensions considered. For instance, in a 3 dimensional space a line of one system (1 dimensional) is linked to a surface of the other (3-1=2 dimensional). In Fig. 1, an example of two complexes of dual cells in a two-dimensional domain is shown. The elements of the dual complex have an orientation which is derived by the primal one. With quadrangular cells, the definition of a system of orthogonal dual grids is easy. As it can be seen, in this case, in fact, each line of the dual grid crosses orthogonally the linked surface of the primal grid.

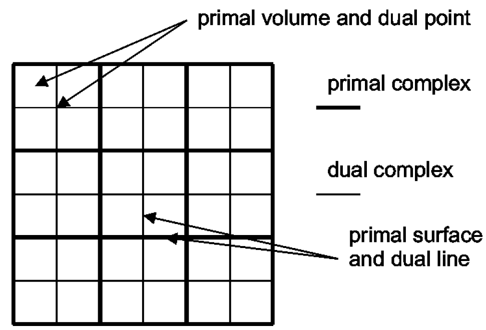

Fig. 1. Primal and dual complex of quadrangolar cells in a two dimensional domain.

As a matter of fact, quadrangular grids are not always well suited for the discretization of the domain of many cases of interest. Simplicial meshes, made by triangles or tetrahedra, are more flexible in dealing with complicated geometrical structures. In this cases the definition of orthogonal grids is more difficult. In two dimensions, an orthogonal system can be defined by means of Delaunay- Voronoi triangularization. Given a primal mesh of triangles, the dual points are set in the triangles circumcenters, each line joining two points of the dual grid is thus orthogonal to the face of the primal one, as shown in Fig. 2.

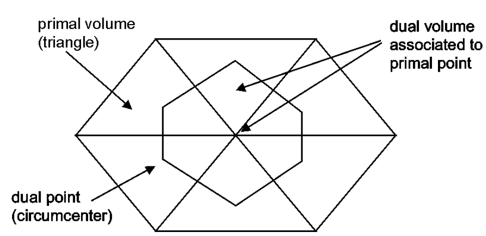

Fig. 2. Primal and dual complex of the Voronoi-Delaunay type on triangular cells.

While the solution of Delaunay-Voronoy polygons can be acceptable in two dimensions, it is unfortunately almost unfeasible in three dimensions, where distorted tetrahedra would lead easily to incongruent situations, as for instance when the circumcenter lies outside the tetrahedron. In this case, the easiest way to associate a point to a tetrahedron is to use its barycenter. However, in this case the line joining two adjacent volumes is no more orthogonal to the face of the other complex.

2.3 Field laws

Global variables obey Maxwell laws in the integral form. In the present case, Gauss and Faraday laws hold. These two laws can be stated in their finite form on the space elements of the discretization. In the electrostatic case without presence of space charge, the finite form of Gauss law applied to a closed surface Ω becomes

$$\sum_i d_i \Psi_i = 0, \tag{1}$$

where index i goes on all the surfaces belonging to Ω, Ψ is the electric flux over the i-th portion of the closed surface and d_i is a coefficient which relates the orientation of the i-th portion of the surface to the orientation of ω so that it can be $+1$ or -1.

Considering a primal mesh of N_V volumes and N_S surfaces, as in network theory, only $N_V - 1$ equations like (1) are linearly independent.

Equation (1) can be easily written for the whole domain in a matrix form

$$[D]\{\Psi\} = 0, \tag{2}$$

where $[D]$ is a matrix whose elements can take values $+1$, -1 or 0 and $\Psi\}$ is the vector of fluxes. Matrix $[D]$ has $N_V - 1$ rows and N_S columns and is clearly analogue to the reduced incidence matrix of network theory.

Faraday law and stationary conditions allow to introduce electric scalar potential, this variable u is associated to the N_V nodes of the dual grid. Again, for gauging the scalar potential, only $N_V - 1$ potential values are unknowns of the problem, unless Dirichlet boundary conditions are applied to constraint the potential values. The definition of the electric scalar potent ial on the dual grid nodes allows to express the electromotive force along an edge of the dual grid as function of the potential values at edge vertices. In fact, along edge i joining nodes k and j, it holds

$$\xi_i = u_j - u_k, \tag{3}$$

in matrix form

$$\{\xi\} = [D]^T\{u\}, \tag{4}$$

where, as said before, by analogy with the network theory, the link between edge electromotive force and nodal potential is given by the transpose of reduced incidence matrix.

2.4 Constitutive laws

Field laws are not sufficient for solving the field problem and constitutive laws or material relations are also needed. Instead of being applied on a pointwise base as in usual field theory, finite formulation requires that the constitutive equations are applied, under the hypothesis of locally uniform field, to a couple of dual entities. For instance in the electrostatic case, the two global variables involved are the electric flux Ψ on the surface of the primal mesh and the electro motive force ξ on the corresponding dual edge. If orthogonality between meshes holds, the relation on the i-th face-edge couple becomes

$$\frac{\Psi}{S_i} = \varepsilon\frac{\xi_i}{\lambda_i}, \tag{5}$$

where S_i is the area of the face, λ_i is the length of the edge and ε is the electric material constant.

In a matrix formulation, equation (5) leads to

$$\{\Psi\} = [G]\{\xi\}, \tag{6}$$

where $[G]$ is a diagonal matrix $N_S \times N_S$. The diagonal structure of the matrix comes from the fact that orthogonality of the two meshes enforces equation (5) which links together only field quantities related to the i-th edge-face couple. By equations (6) and (4) it turns out that

$$\{\Psi\} = [G][D]^T\{u\}. \tag{7}$$

Taking into account equation (7), Gauss law in matrix form becomes:

$$[D]\{\Psi\} = 0 \Rightarrow [D][G][D]^T\{u\} = 0, \tag{8}$$

which leads to a formulation of the field problem analogue to the usual nodal method of circuit theory. Equation (8) represents a system of linear equations where some elements of the electric scalar potential vector $\{u\}$ are known terms coming from non-homogeneous Dirichelet boundary conditions. This fact ensures a nontrivial solution of the system.

2.5 Non-orthogonal grids

As it has been pointed out before, equation (6) holds only in the case of orthogonal grids, otherwise equation (5) would lead to a link between different components of the vector fields, one orthogonal to the face and one along the direction of the edge piercing the face.

The solution of this problem is not univocal and in this application it has been decided to reconstruct the normal component of the electric field on the face by a local interpolation of the potential. This procedure is performed by constructing locally a mesh of tetrahedra over which the electric potential is interpolated. From the gradient of this interpolation a vector electric field is obtained and thus its component normal to the face is obtained. The local interpolation is made by considering all the dual nodes adjacent to the edge-face couple under consideration. In Fig. 3 a schematic view of the considered nodes is presented, for sake of simplicity, a two dimensional case is reported.

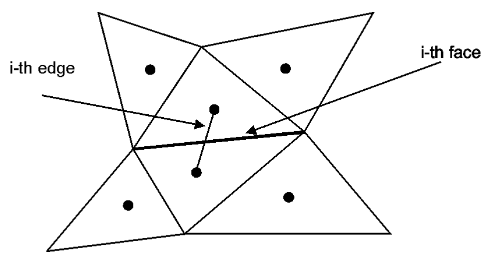

Fig. 3. Nodes involved in the local potential interpolation used to retrieve the normal component of electric field on generic *i*-th face.

Using a local interpolation of potential to get the component of electric field normal to the surface alters the structure of equation (6), this time electric flux on the generic i-th face does not depend only on two potential values, but on a linear combination of more than two nodal values.

Equation (7) becomes

$$\{\Psi\} = [L]\{u\}, \tag{9}$$

where matrix $[L]$, whose dimensions are $N_S \times N_V - 1$, is not anymore diagonal and it is in general non-symmetric.

A final system of equation can anyway be written and its structure is given by

$$[D]\{\Psi\} = 0 \Rightarrow [D][L]\{u\} = 0. \tag{10}$$

Again, known terms obtained by Dirichelet boundary conditions ensures a non-zero solution.

3 Implementation of the Computational Sheme

The scheme outlined in the previous section has been implemented in a three-dimensional code based on a tetrahedral subdivision of the geometric domain. A first grid of tethraedra is firstly set-up and all geometric entities of this mesh are stored with their orientation. Matrix $[D]$ linking together the surface of each tetrahedron to its triangular faces is built. After this phase, dual entities are devised: firstly barycenters of all tetrahedra are computed which are the nodes of the dual grid and afterwards the edges joining the nodes of the dual grid are defined. The orientation of the dual edges is related to the one of the face they pass through. Dual faces and volumes are not built because they are not needed by the algorithm.

Local potential interpolation of equation (9) is performed by means of first order nodal shape functions as of finite element method. Linear interpolation on potential ensures that its gradient is a linear combination of potential values which are the ones most close to the considered face-edge.

Particular conditions are employed for elements lying on the boundaries of the mesh: in case of Dirichelet conditions the known potential value is assigned to the barycenter of the face lying on the boundary, in case of Neumann boundary condition a fictitious node specular to the boundary is generated.

Once matrix equation (10) has been built, it is solved by means of a GMRES algorithm for non-symmetric matrices. This solution gives the values of potential on the nodes of the dual mesh. For post processing reasons it has been decided to store in the case database not these values but the flux ones. In the post-processing phase, electric field is retrieved by means of interpolation on tetrahedra using face elements shape functions. This allows to produce pattern of electric field on lines, maps etc.

4 Numerical Exaples

The electrostatic analysis code has been tested versus several cases with analytical solution and versus electrostatic fields obtained by finite element software on the same tetrahedral mesh.

4.1 Case with uniform field values

This case has been used to assess the capabilities of the proposed algorithm to provide reliable and precise data on distorted meshes and to provide quantitative estimate of the accuracy of the local interpolation proposed. The computational domain is a cube meshed with generic tetrahedra, see Fig. 4. Two Dirichelet boundary conditions on opposite faces are applied in order to produce inside the cube a uniform electric field of 1 V/m. As it can be seen from Fig. 4, even if the domain is very simple, the tetrahedra generated by mesh program do not show an high degree of regularity and this makes the problem very interesting because in this case barycentric dual mesh is far from the orthogonal one. The results obtained have shown a computational accuracy smaller than 1 p.p.m., which is considered very well suited for usual applications and allows to say that the interpolation proposed is working efficiently.

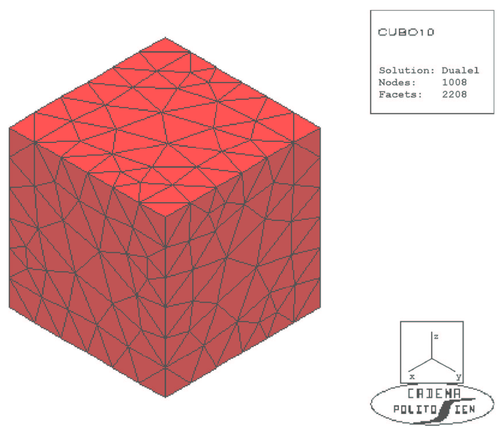

Fig. 4. Computational domain used to assess accuracy of proposed algorithm.

4.2 Cylindrical capacitor

A cylindrical capacitor case has been analysed in order to compare numerical results with the analytical solution. The computational domain is shown in Fig. 5 together with arrows showing the computed electric field. Because of symmetry only a small angular portion of the space between the armatures is meshed.

In Fig. 6 a comparison between analytical electric field and computed one along a radial lines between armatures is presented. As it can be seen

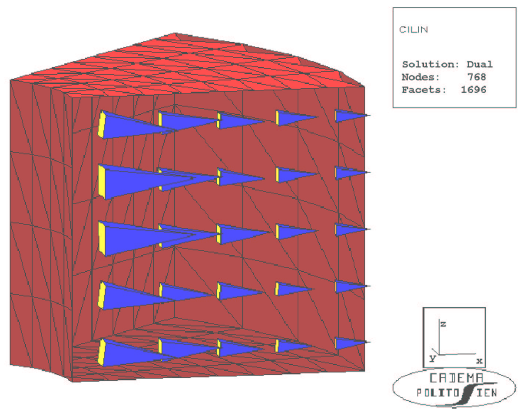

Fig. 5. Cylindrical capacitor with electric field vectors.

from the graph, the agreement between the two patterns is very good even on a case with a relatively coarse mesh as the one presented.

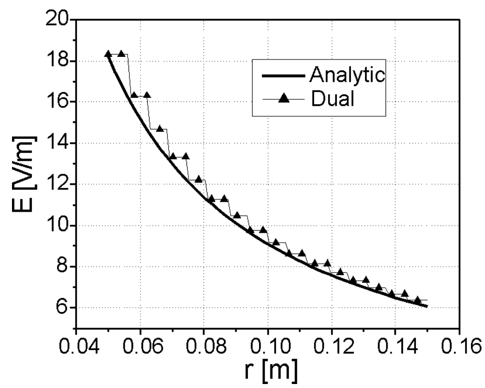

Fig. 6. Comparison between analytical and computed values of electric field along a radial line in the cylindrical capacitor.

4.3 Conductive sphere in a uniform field

An analytical solution is available for the electric field around a conductive sphere immersed in a uniform electric field. In this case the numerical solution has to introduce some discretization errors because of the curvature of the sphere is approximated by means of first order tethraedra and because external electric field is created by means of Dirichlet boundary condition set on opposite mesh walls. Due to the finite extension of the mesh which is not much larger than the sphere for limiting the number of elements, the numerically computed field is sligthly higher than the analytical one. In Fig. 7 the computational domain, 1/8-th of the whole space, is shown together with the electric field arrows. Figure 8 shows the graph of the analytical and numerical behaviour of the electric field along the axis of the structure (direction parallel to the external electric field). As it can be seen, notwithstanding the small overestimation due to the limited extension of the domain, the agreement between the two solutions is more than satisfactory.

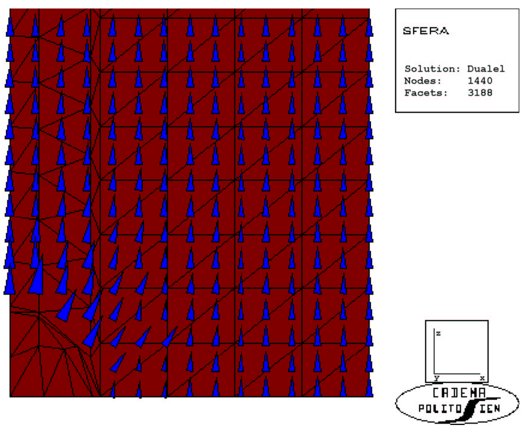

Fig. 7. Conductive sphere in a uniform electric field, the domain discretized is 1/8-th of the whole region.

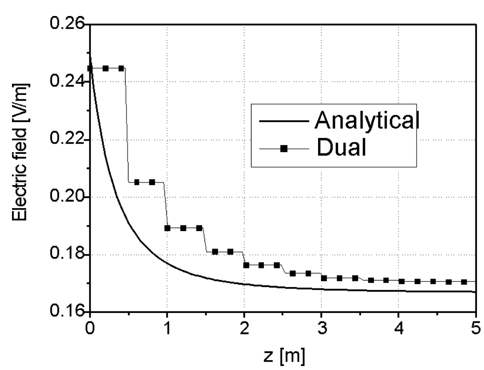

Fig. 8. Comparison between analytical and computed values of electric field along the axis of the sphere problem.

4.4 "L-shaped" domain

This test case has been analysed to assess the accuracy of the proposed procedure versus another numerical field solution. In this case Finite Element

Method with first order tetrahedra was selected and the comparison was performed on the same mesh. In Figure 9 the domain of the problem is presented, it is an "L-shaped" domain with two assigned boundary values. In Figure 10 a comparison between finite element and dual approach computation is presented. The electric field along a line connecting the two corners of the domain is compared and the agreement more than satisfactory. It is worth pointing out that at the inner corner the solution is singular and that, in this case, the new approach is able to give an higher value of electric field.

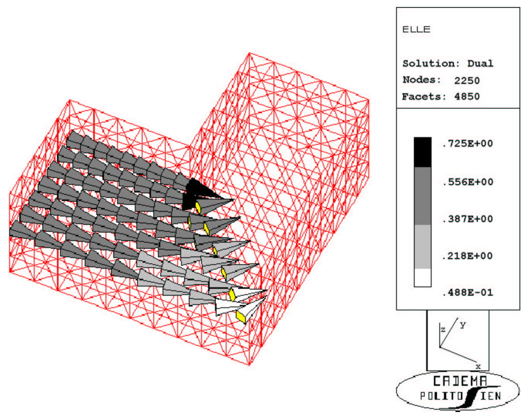

Fig. 9. Electric field vectors inside an "L-saped" domain.

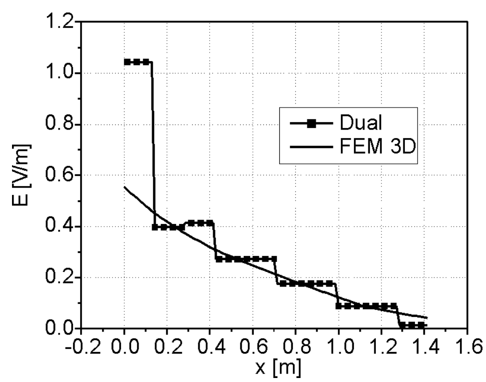

Fig. 10. Comparison between finite element and dual electric field solution on a line joining the two corners of the domain.

5 Conclusion

This paper has presented the results obtained by the numerical implementation of the "finite formulation of electromagnetic field" proposed by Tonti. The results got on a generic three-dimensional grid on electrostatic problems are encouraging and thus the activity on this topic will follow on trying to extend the finite formulation to the coverage of the whole spectrum of electromagnetic problems.

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