

On Pivot Rules for Simplex Method of Interior Points, and their Investigation on Klee-Minty Cube

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Abstract: In [25] it was proposed a parametric linear transformation, which is a "convex" combination of the Gauss transformation of elimination method and the Gram-Schmidt transformation of modified orthogonalization process. Using this transformation, in particular, elimination methods were generalized, Dantzig's optimality criterion and simplex method were developed [26].

The essence of the simplex method development is the following. At each s th step the pivot (positive) vector of length k_s is selected, that allows us to move to improved feasible solution after the step of the generalized Gauss-Jordan complete elimination method. In this method the movement to the optimal point takes place over pseudobases, i.e., over interior points. This method is parametric and finite.

Since the method is parametric there are various variants to choose the pivot vectors (rules), in the sense of their lengths and indices. In this article we propose three rules, which are the development of Dantzig's first rule. These rules are investigated on the Klee-Minty cube (problem) [14, 31]. It is shown that for two rules the number of steps necessary equals to $2n$, and for third rule we obtain the standard simplex method with the largest coefficient rule, i.e., the number of steps for solving this problem is $2^n - 1$.

Keywords: Simplex method of interior points, pseudobasis, Klee-Minty cube, generalized Gauss-Jordan complete elimination method

1 Introduction

In recent years significant research have been dedicated to interior point methods. Over 2000 papers have been published in this area in the last

Manuscript received February 16, 2002

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decade.

For many years after Dantzig's simplex method appearance in 1947 [3], it was unknown whether this method had polynomial complexity. Only in 1972 Klee and Minty showed that the simplex method is not a polynomial algorithm [14]. Namely, they constructed a special class of problems with n variables and $2n$ constraints, and demonstrated that the number of simplex steps necessary equals to $2^n - 1$, if the largest coefficient rule (Dantzig's first rule) is used. The analogous results were also obtained for about all known pricing rules [1, 7, 10, 18].

In 1979 Khachiyan proposed a new polynomial algorithm, based on ellipsoid technique, to solve the linear programming (LP) problem [13]. The ellipsoid technique was proposed by Shor [23] and Nemirovsky and Yudin [15] independently. Even though the ellipsoid method is polynomial, it can not compete with the simplex methods in the practical applications. However, the algorithm not being effective was not critical, because the appearance of an effective algorithm was only a matter of time. Indeed, in 1984 Karmarkar proposed an effective polynomial-time algorithm [12]. In 1986 Barnes [2] demonstrated, that one of the versions of this method coincides with the interior point method, proposed by Dikin in 1967 [4, 5]. However, Dikin didn't consider whether his method was polynomial. Karmarkar's famous paper aroused a lot of interests in using interior point methods for LP. Numerous quicker polynomial algorithms have been obtained in the last years (see, for example, [9, 17, 19, 30, 33]). Thus, the interior point methods direction has undergone numerous developments [21, 24, 32, 33, 34, 35].

In 1980 Tuniev proposed the parametric linear transformation, which is a "convex" combination of the Gauss transformation of elimination method and the Gram-Schmidt transformation of modified orthogonalization process [25]. Using this transformation, in particular, the elimination methods were generalized, a new optimality criterion of the feasible solution for the LP problem was obtained. This criterion is between Dantzig's optimality criterion and Kantorovich's optimality criterion. On this base the simplex method was developed, namely a simplex method of interior points was obtained (see, for example, [26, 27]).

The essence of the simplex method development is the following. At each s th step the pivot (positive) vector of length k_s is selected, that permits to move to the improved feasible solution after the step of the generalized Gauss-Jordan complete elimination method. In this step k_s vectors are included in the pseudobasis and d_s vectors are excluded. In other words, the movement to the optimal point takes place over pseudobases, i.e., over interior points. The objective function value is increased after each step.

The simplex method of interior points is parametric and finite. Here Shor's problem arises: to find a strategy of choice for pivot rules that reaches the optimum by the shortest path. In this paper we propose three rules, which are the development of the largest coefficient rule. We investigate these rules on the Klee-Minty cube [14, 31]. It is shown that for solving this problem the simplex method of interior points using each of two proposed rules requires $2n$ arithmetic operations, whereas using third rule it coincides with Dantzig's simplex method using the largest coefficient rule.

2 Notation

We use the notation proposed by Romanovskii [20]. Here $x[N]$ is the vector $x = \{x_i\}$ with i running over the finite set N ; $x[K]$ is the corresponding K -piece of the vector $x[N]$, where $K \subset N$; $x[i]$ is the component of the vector $x[N]$ with index i .

$z[i, N]$ is the i th row of the matrix $z[M, N] = \{z_{ij}\}$, and $z[M, j]$ is the j th column of this matrix; $z[K, L]$ is a submatrix of the matrix $z[M, N]$, where $K \subset M, L \subset N$.

The matrices $a[M, N]$ and $b[K, L]$ can be multiplied if $N = K$.

$c[M, L] = a[M, N]b[N, L]$ is the product of matrices.

By $o[M, N]$ we denote a matrix with all elements equal to zero (zero matrix), and by $e[N, N]$ an identity matrix. $M = \{1, \dots, m\}$, $N = \{1, \dots, n\}$.

We do not distinguish between row and column vectors.

3 Foundations of Simplex Method of Interior Points

3.1 Generalized Gauss and Gram-Schmidt transformation

Consider the system of vectors $a[1, N], \dots, a[m, N]$. Let $K \subseteq N$ and assume that the Euclidean norm $\|a[i_0, K]\| \neq 0$ for $i_0 \in M$. Set

$$b[i, N] = \begin{cases} \frac{a[i, N]}{\|a[i, K]\|} & \text{if } i = i_0, \\ a[i, N] + \alpha_i b[i_0, N] & \text{otherwise,} \end{cases} \quad (1)$$

where

$$\alpha_i = \alpha_i(K) = -a[i, K]b[i_0, K], \quad i \neq i_0.$$

For an arbitrary $i \neq i_0$ the scalar product

$$b[i, K]b[i_0, K] = (a[i, K] + \alpha_i b[i_0, K])b[i_0, K] = -\alpha_i + \alpha_i = 0,$$

i.e., the subvectors $b[i, K]$ and $b[i_0, K]$ are orthogonal. If $K \subset N$ then the vectors $b[i, N]$ and $b[i_0, N]$ are called partially orthogonal. Note that if $K = N$ then these vectors are orthogonal.

Let $K = \{1, \dots, k\}$, where $k \in \{1, \dots, n\}$. Then the parametric linear transformation (1), on the one hand, is a generalization of the Gauss transformation (for $k > 1$) and, on the other hand, it is a generalization of the Gram-Schmidt transformation (for $k < n$) (see, for example, [25, 26]).

3.2 Generalized elimination methods

Using the generalized Gauss and Gram-Schmidt transformation, in [26] parametric methods for solving the linear equations system

$$a[M, N]x[N] = a[M, 0],$$

where $x[N]$ is an unknown column vector, $m \geq n$, were proposed, which are the generalization of the Gauss elimination and the Gauss-Jordan complete elimination methods. In these methods at each s th step a pivot vector of length k_s is chosen – instead of a pivot element in the elimination methods, where $k_s \in \{1, \dots, n\}$, and all rows of the current system are transformed according to the linear transformation (1). The solution obtained by these methods is unique for any selection of $\{k_s\}$.

If all $k_s = 1$ then the Gauss and Gauss-Jordan elimination methods are obtained.

The generalized Gauss-Jordan complete elimination method allows us to obtain a new decomposition relative to the pseudobasis.

4 Description of Simplex Method of Interior Points

Consider the non-degenerate LP problem presented in canonical form maximize

$$c[N]x[N] \tag{2}$$

subject to

$$a[M, N]x[N] = a[M, 0], \tag{3}$$

$$x[N] \geq o[N], \tag{4}$$

where $x[N]$ is an unknown column vector. Assume that $a[M, M] = e[M, M]$ and $a[M, 0] > o[M]$.

Denote $M' = \{1, \dots, m, m + 1\}$, $N' = \{0, 1, \dots, n\}$ and consider two matrices

$$a[M', N'] = \begin{bmatrix} a[M, N] & a[M, 0] \\ a[m + 1, N] & a[0, 0] \end{bmatrix}$$

and

$$b[M, N] = (e[M, M] \quad o[M, N \setminus M]),$$

where the $(m+1)$ th row of the matrix $a[M', N']$, i.e., the vector $a[m+1, N']$, is the vector of values relative to the initial basis $a[M, 1], \dots, a[M, m]$. $x[N] = b^T[M, N]a[M, 0] = (a[M, 0], o[N \setminus M])$ is the initial basic feasible solution of the problem (2)-(4).

Let us describe the transition from the step s to the step $(s + 1)$, where $s = 1, \dots, s$.

Assume at the s th step the current matrices $d[M', N']$ and $c[M, N]$ be obtained. $d[M', N'] = a[M', N']$ and $c[M, N] = b[M, N]$ at $s = 1$.

1°. If the relative values $d[m + 1, j] \geq 0$ for all $j \in N$, then $x[N] = c^T[M, N]d[M, 0]$ is the optimal feasible solution of the problem (2)-(4).

2°. Let $K = K_1 \cup K_2$ be some subset of the indices of active and non-active columns, that is, $d[m + 1, j] < 0$ for all $j \in K_1$ and $d[m + 1, j] = 0$ for all $j \in K_2$. Let $d[M, 0] > o[M]$ and define

$$\theta_f = \frac{d[f, 0]}{\|d[f, K_f]\|} = \min_i \frac{d[i, 0]}{\|d[i, K_i]\|}$$

for i such that the $d[i, K_i]$ contains all positive components of the vector $d[i, K]$.

3°. The positive pivot vector $d[f, K_f]$ is chosen and the elements of the matrix $d[M', N']$ are transformed according to the step of the generalized Gauss-Jordan complete elimination method, i.e.,

$$d'[i, N'] = \begin{cases} \frac{d[i, N']}{d[i, K_i]} & \text{if } i = f, \\ d[i, N'] + \alpha_i d'[f, N'] & \text{if } i \neq f, \end{cases}$$

where $\alpha_i = -d[i, K_f]d'[f, K_f]$, $i \neq f$.

4°. The matrix $c'[M, N]$ is formed by substituting the row $c'[f, N]$ for $c[f, N]$ in the matrix $c[M, N]$, i.e., the elements of this row are defined as follows:

$$c'[f, j] = \begin{cases} d'[f, j] & \text{for all } j \in K_f, \\ 0 & \text{for all } j \in N \setminus K_f. \end{cases}$$

Thus, the rows of the matrix $c'[M, N]$ are

$$\left. \begin{aligned} c'[f, N] &= \{d'[f, K_f], o[N \setminus K_f]\}, \\ c'[i, N] &= c[i, N] \quad \text{for all } i \in M \setminus f. \end{aligned} \right\}$$

5°. Set $d[M', N'] = d'[M', N']$, $c[M, N] = c'[M, N]$, $s = s + 1$ and go to the step 1°.

Note, that since the number of pseudobases is limited and improved feasible solution obtained at each step is unique, then the simplex method of interior points is finite.

Remark 1. If pivot vector is chosen according to point 2° of the simplex method of interior points then the right-hand side vector of the current system $d[m, 0] \geq o[M]$. Hence, since the current matrix $c[M, N]$ is nonnegative, then $x[N] = c^T[M, N]d[M, 0]$ is the feasible solution of the problem (2)-(4).

Remark 2. If for some $j_0 \in N$ $d[m + 1, j_0] < 0$ and $d[M, j_0] \leq o[M]$, then the problem (2)-(4) is unsolvable.

5 On choice of pivot rule

The simplex method of interior points is parametric, and the path to the optimal point depends on the lengths of the pivot vectors and their indices.

In [29, 28] the following general rule was proposed. At each step direct and reverse passes are organized. The number of the positive components of the feasible improved solution increases at the direct pass, and it decreases at the reverse pass. This general rule can be realized by various ways. In this paper we propose at the direct pass to use one of the following rules, which are the development of the largest coefficient rule.

1. Using the largest coefficient rule, we select the active column, which must not be pseudobasis, and all non-active columns (for example, identity columns).

2. We select all active columns, which must not be pseudobasis, and all non-active columns.

3. We select all active columns, which must not be pseudobasis.

If the number of positive components of the feasible solution does not increase, then we organize reverse pass, selecting only one active column using the largest coefficient rule.

Note that if the length of the pivot vector is equal to 1 at the direct pass, then in this step the corresponding column is not considered.

Let us remark that these rules are possible to modify depending on the class of the problems.

6 Investigation of the pivot rules on the Klee-Minty cube

The Klee-Minty problem in canonical form (Vanderbei's versions) [31] is maximize

$$\sum_{j=1}^n 10^{n-j} x_j, \quad (5)$$

subject to

$$2 \sum_{j=1}^{i-1} 10^{i-j} x_j + x_i + x_{n+i} = \sum_{j=1}^{i-1} 10^{i-j} b_j + b_i, \quad i = 1, \dots, n, \quad (6)$$

$$x_j \geq 0, \quad j = 1, \dots, 2n, \quad (7)$$

where $1 = b_1 \ll b_2 \ll \dots \ll b_n$.

Note that $x = (\underbrace{0, \dots, 0}_n, x_{n+1}, \dots, x_{2n})^T$, where $x_{n+i} = \sum_{j=1}^{i-1} 10^{i-j} b_j + b_i$,

$i = 1, \dots, n$, is the initial basic feasible solution of the problem (5)-(7).

It is known that the simplex method using the largest coefficient rule requires $2^n - 1$ steps to solve the problem (5)-(7) (see, for example [14, 31]). Since the rules 1-3 are the development of the largest coefficient rule, we investigate them on this problem.

Note that in the rules 1 and 2 we consider only identity columns (i.e., one of its components is equal to 1, and the other components are equal to 0) as the non-active columns.

Theorem. *a) Simplex method of interior points using the rule 1 at the direct pass requires $2n$ steps to solve the problem (5)-(7). At the direct pass, by performing n steps, we go from the initial basic feasible solution to the strictly interior point $x[N] = 1/2(b_1, \dots, b_n, b_1, \dots, b_n)^T$, and pass the first half of the path. At the reverse pass, by performing n steps, we go from the strictly interior point to the optimal basic solution, i.e., we pass the second half of the path.*

b) The rules 1 and 2 coincide for the problem (5)-(7).

c) The simplex method of interior points using the rule 3 at the direct pass coincides with the standard simplex method using the largest coefficient rule, i.e., the number of the steps to solve the problem (5)-(7) is $2^n - 1$.

The Theorem is easily proven by the induction method.

Let us describe the outline of the proof for the point *a)* of this theorem.

At the initial step, according to the largest coefficient rule we choose the active column with the number 1. Further, since $b_2 \gg b_1$ then $\theta_1 = \min_i \theta_i$.

As a result, at the first step the pivot vector $d[1, K_1] = (d[1, 1] = 1, d[1, n + 1] = 1)$ of length 2 is chosen. After the step of the generalized Gauss-Jordan complete elimination method we obtain the current system, where right-hand side vector does not contain parameter b_1 , except the first row.

At the second step, according to the largest coefficient rule, the active column with the number 2 is chosen. Since $b_3 \gg b_2$ then $\theta_2 = \min_{i \neq 1} \theta_i$, (θ_1 is not considered since $d[1, K_2] = o[K_2]$). Therefore, at the second step the pivot vector $d[2, K_2] = (d[2, 2] = 1, d[2, n + 2] = 1)$ of length 2 is chosen. After the step of the generalized Gauss-Jordan complete elimination method we obtain the current system, where right-hand side vector does not contain parameter b_2 , except the second row. Continuing in the same way, after n steps we obtain the strictly interior point $x[N] = 1/2(b_1, b_2, \dots, b_n, b_1, b_2, \dots, b_n)$.

At the reverse pass at each step only one active column is selected according to the largest coefficient rule. Note that at the $(n + 1)$ st step the pivot element is the one from n th row of the current table, at the $(n + 2)$ nd step the pivot elements is the one from the $(n - 1)$ th row, and so on. By performing n steps of the reverse pass we go from the strictly interior point to the optimal point.

Let us consider this idea on the problem (5)-(7), where $M = \{1, 2, 3\}$, $N = \{1, \dots, 6\}$. The matrix $d[M', N']$ is represented in the following simplex table.

Table 1. Initial Data.

1	2	3	4	5	6	$d[M, 0]$	α_i
1	0	0	1	0	0	b_1	$:\sqrt{2}$
20	1	0	0	1	0	$10b_1 + b_2$	$-\frac{20}{\sqrt{2}}$
200	20	1	0	0	1	$100b_1 + 10b_2 + b_3$	$-\frac{200}{\sqrt{2}}$
-100	-10	-1	0	0	0	0	$\frac{100}{\sqrt{2}}$

Construct the matrix

$$c[M, N] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = (0, 0, 0, b_1, 10b_1 + b_2, 100b_1 + 10b_2 + b_3)^T$ is the initial basic feasible solution. $K_1 = \{4, 5, 6\}$ is the set of column indices of the initial basis.

Direct pass.

Step 1.

According to the largest coefficient rule, the active column with the number 1 is chosen. Since $b_2 \gg b_1$, then $\theta_1 = \min_{i=1,2,3} \theta_i$. Therefore, at the first step the vector ($d[1, 1] = 1, d[1, 4] = 1$) from Table 1 is the pivot vector. The corresponding coefficients α_i are in the last column of Table 1. After the step of the generalized Gauss-Jordan complete elimination method we obtain the following table.

Table 2. Current Table After First Step.

1	2	3	4	5	6	$d[M, 0]$	α_i
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	0	0	$\frac{b_1}{\sqrt{2}}$	0
10	1	0	-10	1	0	b_2	$:\sqrt{2}$
100	20	1	-100	0	1	$10b_2 + b_3$	$-\frac{20}{\sqrt{2}}$
-50	-10	-1	50	0	0	$50b_1$	$\frac{10}{\sqrt{2}}$

Form the matrix

$$c[M, N] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = (b_1/2, 0, 0, b_1/2, b_2, 10b_2 + b_3)^T$ is the feasible solution. $K_2 = \{1, 4, 5, 6\}$ is the set of column indices of the pseudobasis.

Step 2.

According to the largest coefficient rule, the active column with the number 2 is chosen. Since $b_3 \gg b_2$, then $\theta_2 = \min_{i=2,3} \theta_i$. (Since $\|d[1, K_1]\| = 0$ then θ_1 is not considered). Therefore, at the second step the vector ($d[2, 2] = 1, d[2, 5] = 1$) from Table 2 is the pivot vector. After the step of the generalized Gauss-Jordan complete elimination method we obtain the following table.

Table 3. Current Table After Second Step.

1	2	3	4	5	6	$d[M, 0]$	α_i
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	0	0	$\frac{b_1}{\sqrt{2}}$	0
$\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{b_2}{\sqrt{2}}$	0
0	10	1	0	-10	1	b_3	$:\sqrt{2}$
0	-5	-1	0	5	0	$50b_1 + 5b_2$	$\frac{1}{\sqrt{2}}$

$$c[M, N] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = (b_1/2, b_2/2, 0, b_1/2, b_2/2, b_3)^T$ is the feasible solution. $K_3 = \{1, 2, 4, 5, 6\}$ is the set of column indices of the pseudobasis.

Step 3.

According to the largest coefficient rule, the active column with the number 3 is chosen. Note that since $\|d[1, K_1]\| = \|d[2, K_2]\| = 0$ then $\theta_3 = \min_{i=3} \theta_i$. Therefore, at this step the vector ($d[3, 3] = 1, d[3, 6] = 1$) from Table 3 is the pivot vector. After the step of the generalized Gauss-Jordan complete elimination method we obtain the following table.

Table 4. Current Table After Third Step.

1	2	3	4	5	6	$d[M, 0]$	α_i
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	0	0	$\frac{b_1}{\sqrt{2}}$	0
$\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{b_2}{\sqrt{2}}$	0
0	$\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{b_3}{\sqrt{2}}$	$:\frac{1}{\sqrt{2}}$
0	0	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	$50b_1 + 5b_2 + b_3/2$	$\frac{1}{2}$

$$c[M, N] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = 1/2(b_1, b_2, b_3, b_1, b_2, b_3)^T$ is the feasible strictly interior point. $K_1 = \{1, 2, 3, 4, 5, 6\}$ is the set of column indices of the pseudobasis.

Reverse pass. At the reverse pass we use only the largest coefficient rule.

Step 4.

After the fourth step we obtain the following table.

Table 5. Current Table After Fourth Step.

1	2	3	4	5	6	$d[M, 0]$	α_i
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	0	0	$\frac{b_1}{\sqrt{2}}$	0
$\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-\frac{10}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{b_2}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
0	10	1	0	-10	1	b_3	10
0	5	0	0	-5	1	$50b_1 + 5b_2 + b_3$	5

$$c[M, N] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = (b_1/2, b_2/2, b_3, b_1/2, b_2/2, 0)^T$ is the feasible solution. $K_5 = \{1, 2, 3, 4, 5\}$ is the set of column indices of the pseudobasis.

Step 5.

Table 6. Current Table After Fifth Step.

1	2	3	4	5	6	$d[M, 0]$	α_i
$\frac{1}{\sqrt{2}}$	0	0	$\frac{1}{\sqrt{2}}$	0	0	$\frac{b_1}{\sqrt{2}}$	$:\frac{1}{\sqrt{2}}$
10	1	0	-10	1	0	b_2	10
100	20	1	-100	0	1	$10b_2 + b_3$	100
50	10	0	-50	0	1	$50b_1 + 10b_2 + b_3$	50

$$c[M, N] = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

$x[N] = c^T[M, N]d[M, 0] = (b_1/2, 0, 10b_2 + b_3, b_1/2, b_2, 0)^T$ is the feasible solution. $K_6 = \{1, 3, 4, 5\}$ is the set of column indices of the pseudobasis.

Step 6.

Table 7. Current Table After Sixth Step.

1	2	3	4	5	6	$d[M, 0]$
1	0	0	1	0	0	b_1
20	1	0	0	1	0	$10b_1 + b_2$
200	20	1	0	0	1	$100b_1 + 10b_2 + b_3$
100	10	0	0	0	1	$100b_1 + 10b_2 + b_3$

$$c[M, N] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, after six steps we obtain the optimal feasible basic solution $x[N] = c^T[M, N]d[M, 0] = (0, 0, 100b_1 + 10b_2 + b_3, b_1, 10b_1 + b_2, 0)^T$ of the canonical problem (5)-(7).

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