

The Shannon-Kotelnikov Wavelet in Weighted Spaces

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Abstract: The equivalence between interpolation, uniqueness and basicity in spaces of entire functions is one of the fundamental facts used in investigation of basis sets in weighted spaces of functions. Hilbert spaces of entire functions are naturally mapped onto several weighted Lebesgue spaces without changing the basis properties of a set. This approach makes it possible to use some well known results of the theory of entire functions for investigation of the Shannon-Kotelnikov system in the weighted Lebesgue spaces $L^2(|x|^\omega dx)$.

Keywords: Wavelet, Hilbert space, Shannon-Kotelnikov system.

1 Introduction

In this paper the methods of discrete and continuous harmonic analysis developed by M. M. Djrbashian [1, 2] are used for investigation of some natural wavelet systems in weighted spaces of functions.

It is well known [3, 4, 5, 6] that the function

$$\psi(x) = \frac{2}{\pi(2x-1)} \left\{ \sin 2\pi \left(x - \frac{1}{2} \right) - \sin \pi \left(x - \frac{1}{2} \right) \right\}$$

is a wavelet in the space $L^2(\mathbb{R})$, i.e. the system $\left\{ 2^{k/2} \psi \left(2^k x - l \right) \right\}_{k,l \in \mathbb{Z}}$ is an orthonormal basis in $L^2(\mathbb{R})$. The function $\psi(x)$ is named Shannon-Kotelnikov's wavelet.

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One of the aims of this paper is to show that the mentioned system is a basis in some weighted spaces. We mainly use some well known methods of the theory of entire functions. The new results established in this paper are numerated by numbers while the used well known results are numerated by letters.

2 The spaces $W_\sigma^{p,\omega}$

Definition. Assuming that $p \in (1, +\infty)$, $\omega \in (-1, p - 1)$ and $\sigma \in (0, +\infty)$ are any numbers, we introduce $W_\sigma^{p,\omega}$ as the set of all entire functions $f(z)$ of exponential type $\leq \sigma$, for which

$$\|f\|_{p,\omega} \equiv \|f\|_{W_\sigma^{p,\omega}} = \left\{ \int_{-\infty}^{+\infty} |f(x)|^p |x|^\omega dx \right\}^{1/p} < +\infty. \quad (1)$$

Here and everywhere below we say that an entire function is of exponential type if $\rho = 1$ and $\sigma \in (0, +\infty)$ for its order and type, and we call a representation "descriptive" if the considered class of functions coincides with the set of all functions representable in that form.

The space $W_\sigma^{p,\omega}$ and more general spaces have been introduced by M. M. Džrbashian [1, 2] who found also the descriptive representations of these spaces. The representation of $W_\sigma^{p,\omega}$ is a generalization of the well-known Wiener-Paley theorem since $W_\sigma^{p,0}$ ($\omega = 0$) coincides with their well-known space W_σ^p . Further, if $p = 2$, then $W_\sigma^{p,\omega}$ becomes a Hilbert space with the inner product

$$(f, g) = \int_R f(x) \overline{g(x)} |x|^\omega dx, \quad R = (-\infty, +\infty). \quad (2)$$

We start by some properties of the spaces $W_\sigma^{p,\omega}$, which are proved in [1, 7].

Theorem A. For any $p \in (1, +\infty)$, $\omega \in (-1, p - 1)$ and $\sigma \in (0, +\infty)$:

- 1°. $W_\sigma^{p,\omega}$ is a Banach space.
- 2°. If a sequence $\{f_n\}_1^\infty \subset W_\sigma^{p,\omega}$ converges in the norm (1), then it uniformly converges in any compact set of \mathbb{C} :
- 3°. If $f \in W_\sigma^{p,\omega}$, then

$$e^{i\sigma z} f(z) z^{\omega/p} \in H_+^p \quad \text{and} \quad e^{-i\sigma z} f(z) z^{\omega/p} \in H_-^p, \quad (3)$$

where H_{\pm}^p are the ordinary Hardy spaces over the upper and lower half-planes.

3°. If $f \in W_{\sigma}^{p,\omega}$, then for any $z = x + iy \in \mathbb{C}$

$$|f(z)| \leq A \|f\|_{p,\omega} e^{\sigma|y|} (1 + |z|)^{-\omega/p} (1 + |y|)^{-1/p}, \tag{4}$$

where A is a constant independent of f and z .

We recall that the Mittag-Leffler type function

$$E_{\rho}(z, \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \quad \rho > 0, \quad \mu \in \mathbb{C}, \tag{5}$$

is an entire function of order ρ and type $\sigma = 1$. This function has been investigated in [1, 2] and has many applications. The main fields of these application are described in detail in [1] containing a large reference list. One can see that for particular values of ρ and μ

$$\begin{aligned} E_1(z, 1) &= e^z, & E_1(z, 2) &= \frac{e^z - 1}{z} \\ E_{1/2}(z, 1) &= \cosh \sqrt{z}, & E_{1/2}(z, 2) &= \frac{\sinh \sqrt{z}}{\sqrt{z}}. \end{aligned} \tag{6}$$

We will also use the following Wiener-Paley type theorem on descriptive representation of $W_{\sigma}^{p,\omega}$, that is a special case of the general result proved in [1, 2].

Theorem B. 1°. The class $W_{\sigma}^{2,\omega}$ ($-1 < \omega < 1$) coincides with the set of all functions representable in the form

$$f(z) = \int_{-\sigma}^{\sigma} E_1(i\tau z, \mu) |\tau|^{\mu-1} \varphi(\tau) d\tau, \quad z \in \mathbb{C}, \tag{7}$$

where $\mu = 1 + \omega/2$ and $\varphi(\tau)$ is any function of $L^2(-\sigma, \sigma)$.

2°. For any $f \in W_{\sigma}^{2,\omega}$ ($-1 < \omega < 1$) the following inversion formula is true:

$$\begin{aligned} \frac{1}{2\pi} \frac{d}{d\tau} \int_R \frac{e^{-it\tau} - 1}{-it} \exp\left\{i\frac{\pi}{2}(\mu - 1) \operatorname{sign}(t\tau)\right\} f(t) |t|^{\mu-1} dt = \\ = \begin{cases} \varphi(\tau) & \text{for } \tau \in (-\sigma, \sigma) \\ 0 & \text{for } \tau \notin (-\sigma, \sigma). \end{cases} \end{aligned} \tag{8}$$

3°. For some constants $a, b > 0$ which are independent of f and φ ,

$$a\|\varphi\|_2 \leq \|f\|_{2,\omega} \leq b\|\varphi\|_2, \tag{9}$$

where $\|\varphi\|_2 = \left[\int_{-\sigma}^{\sigma} |\varphi(\tau)|^2 d\tau \right]^{1/2}$.

Since $E_1(iz\tau, 1) = e^{izz\tau}$, Theorem B contains Winer-Paley theorem in the particular case $\omega = 0, \mu = 1$. On the other hand, for $W_{\sigma}^{2,0} \equiv W_{\sigma}^2$ the following Shannon-Kotelnikov interpolation theorem is true.

Theorem C. For any sequence $\{a_k\} \in l^p$ ($1 < p < +\infty$) the series

$$f(z) = \sum_{-\infty}^{\infty} (-1)^k a_k \frac{\sin \sigma z}{\sigma \left(z - \frac{\pi k}{\sigma} \right)}, \quad \sigma > 0, \tag{10}$$

uniformly converges in any disc $|z| \leq R$. Besides, this series converges in the norm of $L^p(R)$ and defines a continuously invertible operator which maps l^p to W_{σ}^p , and for some constants $C_{1,2} > 0$ independent of $f(z)$

$$C_1 \|f\|_p \leq \|\{a_k\}\|_p \leq C_2 \|f\|_p,$$

where $a_k = f\left(\frac{\pi k}{\sigma}\right)$ and $\|\{a_k\}\|_p = \sum_k |a_k|^p$.

The same assertion can be stated otherwise.

Theorem C'. The following system is a Riesz basis in W_{σ}^p :

$$\left\{ (-1)^k \frac{\sin \sigma z}{\sigma \left(z - \frac{\pi k}{\sigma} \right)} = \frac{\sin \sigma \left(z - \frac{\pi k}{\sigma} \right)}{\sigma \left(z - \frac{\pi k}{\sigma} \right)} \right\}_{k=-\infty}^{\infty}. \tag{11}$$

Recall that a Riesz basis is one satisfying the inequalities (10). If $p = 2$, then (10) becomes the Parseval equality $\|\{a_k\}\|_2 = \|f\|_2$.

A Shannon-Kotelnikov type theorem is true also for the general spaces $W_{\sigma}^{p,\omega}$ [3]. Defining $l^{p,\omega}$ as the space of all sequences for which

$$\|\{a_k\}\|_{p,\omega}^p = \sum_{k=-\infty}^{\infty} |a_k|^p (1 + |k|)^{\omega} < +\infty, \tag{12}$$

one can formulate that theorem as follows.

Theorem D. Let $\{a_k\} \in l^{p,\omega}$ ($1 < p < +\infty, -1 < \omega < 1$) be an arbitrary sequence. Then the series (9) uniformly converges in any disc $|z| \leq R$. Besides, this series converges in the norm of $W_{\sigma}^{p,\omega}$ and defines a continuously

invertible operator which maps $l^{p,\omega}$ to $W_\sigma^{p,\omega}$, and for some constants $C'_{1,2} > 0$ independent of $f(z)$

$$C'_1 \|f\|_{p,\omega} \leq \|\{a_k\}\|_{p,\omega} \leq C'_2 \|f\|_{p,\omega},$$

where $a_k = f(\frac{\pi}{\sigma}k)$.

For $p = 2$ this assertion takes the following form.

Theorem D'. *The system (11) is a Riesz basis in $W_\sigma^{2,\omega}$ ($-1 < \omega < 1$).*

In contrast to the case $\omega = 0$, the set (11) is not orthonormal in Theorem D' (where $\omega \neq 0$) but one can write its byorthogonal set in an explicit form.

3 Bases in weighted spaces $L^{2,\omega}(R)$

The space $L^{p,\omega}(R)$ ($1 < p < +\infty$, $-1 < \omega < 1$) is the set of all functions which are measurable on $R = (-\infty, +\infty)$ and such that

$$\|f\|_{p,\omega} \equiv \|f\|_{L^{p,\omega}(R)} = \left\{ \int_R |f(x)|^p |x|^\omega dx \right\}^{1/p} < +\infty. \quad (14)$$

For $p = 2$ this Banach space becomes a Hilbert space with the inner product

$$(f, g) = \int_R f(x) \overline{g(x)} |x|^\omega dx. \quad (15)$$

Assuming that $\sigma > 0$ and $-1 < \omega < 1$, we denote $V_0^\omega = W_\sigma^{2,\omega}$ and $V_k^\omega = W_{2^k\sigma}^{2,\omega}$, where $k \in Z$ ($Z = 0, \pm 1, \pm 3, \dots$) and $W_\sigma^{2,\omega}$ is the weighted space of entire functions with the norm (1), which is defined in Sec. 1. One can observe that $W_{2^k\sigma}^{2,\omega} \subset L^{2,\omega}(R)$.

Lemma 1. *The sequence of subspaces $\{V_k^\omega = W_{2^k\sigma}^{2,\omega}\}_{k \in Z}$ forms a multiresolution analysis (MRA) in $L^{2,\omega}(R)$, i.e. it satisfies the following requirements:*

- (i) $V_k^\omega \subset V_{k+1}$,
- (ii) $\overline{\bigcup_{k \in Z} V_k^\omega} = L^{2,\omega}(R)$,
- (iii) $\overline{\bigcap_{k \in Z} V_k^\omega} = \{0\}$,
- (iv) $f(x) \in V_k^\omega$ if and only if $f(2x) \in V_{k+1}^\omega$,
- (v) There exists a function $\Phi \in V_0$ for which the set $\{\Phi(x - \frac{\pi}{\sigma}l)\}_{l \in Z}$ is a Riesz basis in V_0^ω .

Proof. (i) and (iv) directly follow from the definition of $W_\sigma^{2,\omega}$, while (v) is a consequence of Theorem D' by which $\left\{ \frac{\sin \sigma(x - \frac{\pi}{\sigma}l)}{\sigma(x - \frac{\pi}{\sigma}l)} \right\}_{l \in Z}$ is a Riesz basis in $W_\sigma^{2,\omega}$. For proving (ii) (i.e., showing that $V_k^\omega \rightarrow L^{2,\omega}(R)$ as $k \rightarrow +\infty$), we suppose $F(x)$ to be any function of $L^{2,\omega}(R)$ and construct a sequence $\{f_k\}_{k \in Z}$ such that $f_k \in V_k^\omega$ and $\|F - f_k\|_{2,\omega} \rightarrow 0$ as $k \rightarrow +\infty$. We use the following theorem which is the case $\rho = 1$ of a general result of M. M. Djrbashian [1, 2].

Theorem (M. M. Djrbashian) *Let $-1 < \omega < 1$. Then $F(x) \in L^{2,\omega}(R)$ if and only if this function is representable in the form*

$$F(x) = \int_R E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} h(\tau) d\tau, \tag{16}$$

where $h \in L^2(R)$. If $F(x) \in L^{2,\omega}(R)$, then the integral (16) is convergent in the norm of $L^{2,\omega}(R)$ and the following inversion formula is true:

$$\frac{1}{2\pi} \frac{d}{d\tau} \int_R \frac{e^{ix\tau} - 1}{ix} f(x) \left(e^{-i\frac{\pi}{2} \text{sign}(x\tau)} |x| \right)^{\omega/2} dx = h(\tau) \tag{17}$$

for almost all $\tau \in R$. Besides,

$$a \|h\|_2 \leq \|f\|_{2,\omega} \leq b \|h\|_2. \tag{18}$$

for some constants $a, b > 0$ depending solely on ω . (One can see that this becomes the well known Plancherel's theorem in the case $\omega = 0$).

Returning to the proof of Lemma 1, observe that by the above theorem for any $F(x) \in L^{2,\omega}$ there exists a function $\varphi(\tau) \in L^2(R)$ by which $F(x)$ is written in the form (16), i.e.

$$F(x) = \int_R E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi(\tau) d\tau. \tag{19}$$

Introducing the functions

$$\varphi_k(x) = \begin{cases} \varphi(x) & \text{if } x \in (2^k\sigma, 2^{k+1}\sigma) \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_k(x) = \int_{-2^k\sigma}^{2^k\sigma} E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi_k(\tau) d\tau, \tag{20}$$

we conclude that $f_k(z) \in W_{2^k\sigma}^{2,\omega} = V_k^\omega$ by Theorem B. Further, by (19) and (20)

$$\begin{aligned} F(x) - f_k(x) &= \int_{R/(-2^k\sigma, 2^k\sigma)} E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} [\varphi(\tau) - \varphi_k(\tau)] d\tau \\ &= \int_{R/(-2^k\sigma, 2^k\sigma)} E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi(\tau) d\tau. \end{aligned} \tag{21}$$

On the other hand, by (18)

$$\|F - f_k\|_{p,\omega} = \int_R |F(x) - f_k(x)|^2 |x|^\omega dx \leq b \int_{R/(-2^k\sigma, 2^k\sigma)} |\varphi(\tau)|^2 d\tau,$$

where the right-hand side integral tends to zero as $k \rightarrow +\infty$.

For proving (iii), observe that if $f(z) \in \bigcap_{k \in Z} V_k^\omega$, then $f \in W_{2^k\sigma}^{2,\omega}$ for any $k \in Z$. If $k = -n < 0$, then $2^k\sigma = \sigma/2^n \rightarrow 0$ as $n \rightarrow +\infty$. Hence, the entire function $f(z)$ is of type zero. On the other hand, $f(x) \in L^{2,\omega}(R)$. Therefore $f(z)$ is representable in the form (7), where $\sigma = 0$. Hence $f(z) \equiv 0$ or what is the same $\bigcap_k V_k^\omega = \{0\}$. This completes the proof of Lemma 1.

Let A_k^ω be the orthogonal complement of V_k^ω to V_{k+1}^ω , i.e. $V_{k+1}^\omega = V_k^\omega \oplus A_k^\omega$, $k \in Z$. Then for any function $f \in V_{k+1}^\omega$ there exist unique $f_1 \in V_k^\omega$ and $f_2 \in A_k^\omega$ such that $f(z) = f_1(z) + f_2(z)$ and $(f_1, f_2) = 0$. On the other hand, it is obvious that for any $j \in Z$ $V_{k+1}^\omega = A_k^\omega \oplus V_k^\omega = A_k^\omega \oplus A_{k-1}^\omega \oplus V_{k-1}^\omega = \dots = \left[\bigoplus_{j=0}^m A_{k-j}^\omega \right] \oplus V_{k-m}^\omega = \bigoplus_{j \leq k} A_j^\omega$. Here the sequence of spaces V_k^ω is defined by means of V_0^ω . Hence all spaces A_k^ω are defined by A_0^ω which is the orthogonal complement of V_0^ω to V_1^ω , i.e. $V_1^\omega = V_0^\omega \oplus A_0^\omega$. Now we shall describe the spaces A_0^ω .

Lemma 2. *The space A_0^ω coincides with the set of functions $\psi(x)$ of the form*

$$\begin{aligned} \psi(x) &= \int_{-2\sigma}^{-\sigma} E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi_1(\tau) d\tau + \\ &= \int_{\sigma}^{2\sigma} E_1(-ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi_2(\tau) d\tau, \end{aligned} \tag{22}$$

where $\varphi_1 \in L^2(-2\sigma, -\sigma)$ and $\varphi_2 \in L^2(\sigma, 2\sigma)$.

Proof. Since A_0^ω which is the orthogonal complement of V_0^ω to V_1^ω , it suffices to show that

$$A_0^\omega \subset V_1^\omega \quad \text{and} \quad A_0^\omega \perp V_0^\omega.$$

If $\psi(x) \in A_0^\omega \subset V_1^\omega = W_{2\sigma}^{2,\omega}$, then by Theorem B

$$\psi(x) = \int_{-2\sigma}^{2\sigma} E_1(ix\tau, 1 + \omega/2) |\tau|^{\omega/2} \varphi(\tau) d\tau, \quad (23)$$

where $\varphi(\tau) \in L^2(-2\sigma, 2\sigma)$. On the other hand, $\psi(x) \perp V_0^\omega = W_\sigma^{2,\omega}$ by our definitions and the set (11) is a Riesz basis in $W_\sigma^{2,\omega}$ by Theorem D'. Hence, it suffices to show that $\psi(x)$ is orthogonal to the set (11), i.e.

$$\int_R \psi(x) \frac{\sin \sigma \left(x - \frac{\pi k}{\sigma}\right)}{\sigma \left(x - \frac{\pi k}{\sigma}\right)} |x|^\omega dx = 0, \quad k \in Z. \quad (24)$$

To this end, we shall use the simple equality

$$\frac{\sin \sigma \left(x - \frac{\pi k}{\sigma}\right)}{\sigma \left(x - \frac{\pi k}{\sigma}\right)} = \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} e^{i\tau x} e^{-i\frac{\pi k}{\sigma}\tau} d\tau \quad (25)$$

and the following generalization of Parseval equality for the representations (23) and (25) (see [2], p. 250):

$$\int_R \psi(x) \frac{\sin \sigma \left(x - \frac{\pi k}{\sigma}\right)}{\sigma \left(x - \frac{\pi k}{\sigma}\right)} |x|^\omega dx = \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \varphi(\tau) e^{-i\frac{\pi k}{\sigma}\tau} d\tau. \quad (26)$$

Hence by (24)

$$\frac{1}{2\sigma} \int_{-\sigma}^{\sigma} \varphi(\tau) e^{-i\frac{\pi k}{\sigma}\tau} d\tau = 0, \quad k \in Z.$$

The system $\left\{e^{-i\frac{\pi k}{\sigma}\tau}\right\}_{k \in Z}$ is complete in $L^2(-\sigma, \sigma)$. Therefore $\varphi(\tau) = 0$ for almost all $\tau \in (-\sigma, \sigma)$ and (22) follows from (23). Thus, Lemma 2 is proved.

If the set $\left\{\psi\left(x - \frac{\pi k}{\sigma}\right)\right\}_{k \in Z}$ is a Riesz basis in A_0^ω for a function $\psi(x) \in A_0^\omega$, then obviously the set $\left\{\psi\left(2x - \frac{\pi k}{\sigma}\right)\right\}_{k \in Z}$ is a Riesz basis in A_1^ω and the set $\left\{\psi\left(2^n x - \frac{\pi k}{\sigma}\right)\right\}_{k \in Z}$ is a Riesz basis in A_n^ω for any $n \geq 0$. Therefore the united set $\left\{\psi\left(2x - \frac{\pi k}{\sigma}\right)\right\}_{n, k \in Z}$ is a Riesz basis in $L^{2,\omega}(R)$ by Lemma 1. In

this point of view, it is reasonable to describe all functions $\varphi_1(\tau)$ and $\varphi_2(\tau)$ in (22), for which $\left\{ \psi \left(2^n x - \frac{\pi k}{\sigma} \right) \right\}_{k \in Z}$ is a Riesz basis. Intending to consider this general question in another paper, here we shall deal with its particular case.

We introduce the entire function

$$g(z) = \frac{e^{i2\sigma z} - e^{i\sigma z}}{2iz\sigma}, \quad \sigma > 0. \tag{27}$$

It is easy to see that $|g(z)| \leq \frac{M}{1+|z|} e^{2\sigma|y|}$ ($z = x + iy \in \mathbb{C}$), i.e. $g(z)$ is of the order $\rho = 1$ and of the type 2σ . On the other hand, it is evident that

$$\int_R |g(x)|^2 |x|^\omega dx \leq M \int_R \frac{|x|^\omega}{(1+|x|)^2} dx < +\infty, \quad -1 < \omega < 1.$$

Thus $g(x) \in L^2(R)$ and $g(z) \in W_{2\sigma}^{2,\omega}$. For finding the type of $g(z)$ in the upper and lower half-planes, observe that

$$|g(z)| \leq \frac{e^{-2\sigma y} + e^{-\sigma y}}{2|z|\sigma}, \quad z = x + iy \in \mathbb{C}.$$

Hence

$$\begin{aligned} h_g \left(\pm \frac{\pi}{2} \right) &= \limsup_{y \rightarrow +\infty} |y|^{-1} \ln |g(iy)| \\ &= \limsup_{y \rightarrow +\infty} |y|^{-1} [\ln(e^{-2\sigma y} + e^{-\sigma y}) - \ln 2|y|\sigma], \end{aligned}$$

for the indicator of $g(z)$. Therefore, $h_g(\pi/2) = -\sigma$ and $h_g(-\pi/2) = -2\sigma$ since $e^{-2\sigma y} + e^{-\sigma y} \sim e^{-\sigma y}$ as $y \rightarrow +\infty$ and $e^{-2\sigma y} + e^{-\sigma y} \sim e^{-2\sigma y}$ as $y \rightarrow -\infty$. Hence, the indicator diagram of $g(z)$ is the segment $[-2\sigma i, -\sigma i]$ of imaginary axis. Consequently, the conjugate diagram of $g(z) \in W_{2\sigma}^{2,\omega}$ is the segment $[i\sigma, i2\sigma]$. By a small shift in Theorem B we conclude that there exists a function $\varphi_2(\tau) \in L^2(\sigma, 2\sigma)$ such that

$$\frac{e^{i2\sigma z} - e^{i\sigma z}}{2iz\sigma} = \int_\sigma^{2\sigma} E_1(i\tau z, 1 + \omega/2) |\tau|^{\omega/2} \varphi_2(\tau) d\tau. \tag{28}$$

Repeating the above arguments for the entire function

$$g_1(z) = \frac{e^{-i\sigma z} - e^{-2i\sigma z}}{iz\sigma}$$

we conclude that its order is $\rho = 1$, its type is 2σ and its conjugate diagram coincides with $[-2i\sigma, -i\sigma]$. Using Theorem B one more time, we conclude that there exists a function $\varphi(\tau) \in L^2(-2\sigma, -\sigma)$ such that

$$g_1(z) = \frac{e^{-i\sigma z} - e^{-2i\sigma z}}{2iz\sigma} = \int_{-2\sigma}^{-\sigma} E_1(i\tau z, 1 + \omega/2) |\tau|^{\omega/2} \varphi_1(\tau) d\tau. \tag{29}$$

The inversion formula (8) permits to write down the functions $\varphi_{1,2}(\tau)$ explicitly:

$$\varphi_1(\tau) = \frac{1}{2\pi} \frac{d}{d\tau} \int_R \frac{e^{-it\tau} - 1}{-it} \exp \left\{ i \frac{\pi}{4} (\mu - 1) \text{sign}(t\tau) \right\} g(t) |t|^{\omega/2} dt,$$

$$\varphi_2(\tau) = \frac{1}{2\pi} \frac{d}{d\tau} \int_R \frac{e^{-it\tau} - 1}{-it} \exp \left\{ i \frac{\pi}{4} (\mu - 1) \text{sign}(t\tau) \right\} g_1(t) |t|^{\omega/2} dt.$$

Inserting these formulas into (22) and using (28) and (29) we find that

$$\psi(z) = \frac{e^{-i\sigma z} - e^{-2i\sigma z} + e^{2i\sigma z} - e^{i\sigma z}}{2iz\sigma} = \frac{\sin 2\sigma z - \sin \sigma z}{\sigma z}. \tag{30}$$

The above argument remains true if we replace z by $z + \alpha$ in $g(z)$ and $g_1(z)$ (where $\alpha \in R$ is any fixed number). This leads to the following formula:

$$\psi_\alpha(z) = \frac{\sin 2\sigma(z + \alpha) - \sin \sigma(z + \alpha)}{\sigma(z + \alpha)}. \tag{31}$$

Summarizing all this with Theorem D', we come to the proof of

Theorem 1. *The following system is a Riesz basis in A_1^ω :*

$$\left\{ \psi \left(z - \frac{\pi}{\sigma} k \right) = \frac{\sin 2\sigma \left(z + \alpha - \frac{\pi}{\sigma} k \right) - \sin \sigma \left(z + \alpha - \frac{\pi}{\sigma} k \right)}{\sigma \left(z + \alpha - \frac{\pi}{\sigma} k \right)} \right\}_{k \in Z}. \tag{32}$$

This result can be stated also in the form of

Theorem 2. *The following system is a Riesz basis in $L^{2,\omega}(R)$:*

$$\left\{ \psi \left(2^n x + \alpha - \frac{\pi}{\sigma} k \right) \right\}_{n,k \in Z}. \tag{33}$$

Taking $\sigma = \pi$ in the above arguments we come to the system (33), where the shifts are only by entire numbers, and hence $\psi_\alpha(x) = \psi(x + \alpha)$ becomes

a wavelet which is not orthogonal but is *byorthogonal*. It is well known that the system (33) is an orthonormal basis in $L^2(R)$. This basis arises from (31) which is known as Shannon-Kotelnikov wavelet. Thus, we came to

Theorem 3. *The Shannon-Kotelnikov function*

$$\psi_\alpha(x) = \frac{\sin 2\pi(x + \alpha) - \sin \pi(x + \alpha)}{\pi(x + \alpha)}$$

is a *byorthogonal wavelet* in $L^{2,\omega}(R)$, whatever be $\omega \in (-1, 1)$.

One can observe the interesting fact that the convergence character of the series (3.10) passes to the Shannon-Kotelnikov's system.

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