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FIR NOTCH FILTER DESIGN – A REVIEW

Suhash Chandra Dutta Roy, Balbir Kumar and Shail Bala Jain

(Invited Paper)

Abstract: Notch filters are invariably used in communication, control, instrumentation, and bio-medical engineering, besides a host of other fields, to eliminate noise and power line interferences. Digital notch filters can be designed as infinite impulse response (IIR) as well as finite impulse response (FIR) structures. As compared to the latter, IIR filters have the advantage that they require lower orders for efficient approximation of a given set of specifications. However, IIR filters are potentially unstable and do not provide linear phase characteristics, in general. FIR filters, on the other hand, are unconditionally stable and can be designed to give exact linear phase characteristics. We, in this review paper, focus our attention to the recent design techniques proposed by us for FIR notch filters.

Standard FIR filter design methods, such as windowing, frequency sampling and computer-aided/optimization may be used for designing FIR notch filters. However, most of these methods result in ripples in the passbands. In many situations, maximally flat (MF) filters are preferred since they have maximum attenuation in the stopband and hence can yield the best signal-to-noise ratio. A number of methods are available in the literature for designing MF digital filters. We, in this paper, review the design techniques for computing the weights of MF FIR notch filters. A number of design methodologies have been highlighted that lead to either recursive or explicit formulas for the computation of weights of FIR notch filters.

Procedures for the design of FIR notch filters with maximal flatness of the amplitude response (in the Butterworth sense) at $\omega = 0$ and $\omega = \pi$ have

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S. C. Dutta Roy is with Department of Electrical Engineering, Indian Institute of Technology (I.I.T. Delhi) Hauz Khas, New Delhi-110016 (e-mail: scdroy@ee.iitd.ernet.in). B. Kumar is with Department of Electronics and Communication Engineering, Netaji Subhas Institute of Technology (NSIT), Sector-3, Dwarka, New Delhi-110045. S.B. Jain is with Department of Electronics and Communication Engineering, Delhi College of Engineering (DCE), Bawana Road, New Delhi-110042 (e-mail: ishank@bol.net.in).

been given. Empirical formulas for finding the filter length N have also been proposed. By relaxing the linear phase property, it is possible to reduce the filter order required for a given magnitude response specifications. An FIR filter (with non-linear phase) can be derived from a second order IIR notch filter prototype. Explicit mathematical formulas for computing the weights for such FIR notch filters have been given. Design approaches based on the use of (i) Bernstein polynomials, and (ii) lowpass filter design have also been exploited to obtain maximally flat FIR notch filters.

Key words: Digital filters, FIR filters, notch filters.

1. Introduction

1.1 Notch filters

Digital signal processing (DSP) techniques have rapidly developed in the recent years due to advances in digital computer technology and integrated circuit fabrication [3], [26], [27]. The use of digital circuits yields high speed as well as high reliability, and also permits us to have programmable operations. DSP techniques find applications in a variety of areas such as speech processing, data transmission on telephone channels, image processing, instrumentation, bio-medical engineering, seismology, oil exploration, detection of nuclear explosion, and in the processing of signals received from the outer space, besides others. Various types of digital filters, such as Low-pass (LP), High-pass (HP), Band-pass (BP), Band-stop (BS), and Notch filters (NF), and various types of digital operations such as Differentiation, Integration and Hilbert transformation, to mention a few, are invariably used in many of the applications just mentioned. In this review paper, we focus our attention on the design and performance analysis of notch filters.

1.1.1 Notch filter characteristics

A notch filter highly attenuates/ eliminates a particular frequency component from the input signal spectrum while leaving the amplitude of the other frequencies relatively unchanged. A notch filter is, thus, essentially a bandstop filter with a very narrow stopband and two passbands. The amplitude response, $H_1(\omega)$, of a typical notch filter (designated as NF1) is shown in Fig. 1 and is characterized by the notch frequency, ω_d (in radians) and 3-dB rejection bandwidth, $\overline{BW}$. For an ideal notch filter, $\overline{BW}$ should be zero, the passband magnitude should be unity (zero dB) and the attenuation at the notch frequency should be infinite.

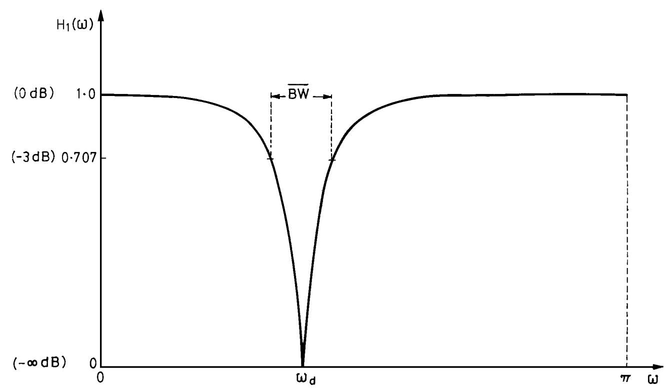

Fig. 1. The amplitude response $H_1(\omega)$ of notch filter: NF1.

We may, alternatively, have the amplitude response, $H_2(\omega)$, of a notch filter (designated as NF2) as shown in Fig. 2. $H_2(\omega)$ has 180 degrees phase shift beyond the notch frequency ω_d. However, the magnitude response $|H_2(\omega)|$ is of the same type as that shown in Fig. 1. We review methodologies for approximating notch filters of both the types.

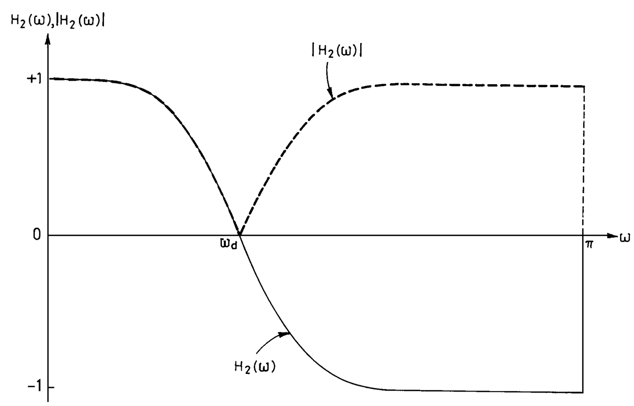

Fig. 2. The response $H_2(\omega)$ and $|H_2(\omega)|$ of notch filter: NF2.

1.2 Digital notch filter design techniques

Digital Notch filters may be designed as infinite impulse response or finite impulse response structures by using standard design techniques. The salient features of these techniques with specific reference to notch filters will be briefly discussed.

1.2.1 IIR designs

In situations where linearity of phase is not important, IIR filters are preferred since these require much lower order than the FIR ones for the same set of magnitude response specifications. The commonly used IIR filter design methods require transforming the given specifications to an equivalent analog filter (by using bilinear transformation, for example). We then design the analog notch filter and finally convert it back to the digital domain through inverse transformation. This approach has the advantage that the standard results of analog filter design can be conveniently used. Based upon this approach, one may design Butterworth, Chebyshev, or elliptic filters [24], [26]. Besides these, IIR notch filters may also be designed by using Padé approximation, least-squares approach or filter parameter optimization techniques.

Some modified designs, specifically for IIR notch filters, are also available. Hirano et al. [12] have realized IIR notch filter function by applying bilinear transformation on second order analog transfer function. The design requires only two multipliers and offers independent variation of notch frequency (ω_d) and the 3-dB rejection bandwidth ($\overline{BW}$). Laakso et al. [19] have proposed first and second order IIR notch filters with zeros strictly on the unit circle and poles close to the zeros to ensure a narrow notch width. The second order notch filter given by

$$H(z) = K\frac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}, \tag{1}$$

can be designed for an arbitrary notch frequency ω_0. In (1), r is the radius of the complex conjugate pole pair located at the frequency ω_0 and K is a scaling factor. In this design, $\overline{BW}$ can be controlled through r, being narrower as r goes closer to the unit circle [19]. However, the quantization error increases when $\Delta = 1 - |r|$ is made small (since the variance of the quantization error is proportional to $1/\Delta^2$ [26]).

In certain applications of signal processing, where it is desired to eliminate unknown or time-varying narrow-band or sine-wave components from the observed time series, we prefer an adaptive notch filter (ANF). Adaptive notch filter designs have been proposed by Thompson [34], Rao and Kung [29], Friedlander and Smith [8], and Nehorai [21], amongst others. The computational efficiency, stability, convergence and numerical robustness of these methods depend upon the algorithm used for adaptation.

One of the major problems in IIR filters is that these designs have nonlinear phase response and, therefore, introduce phase distortion in general.

However, it is possible to reduce phase distortion by cascading an all-pass phase equalizer.

We now examine some of the design techniques used for FIR notch filters.

1.2.2 FIR designs

There are essentially three well known classes of design techniques for linear phase FIR filters, namely: frequency sampling, windowing, and optimal (in the Chebyshev sense) design. Frequency sampling method is often not used for notch filter design because the desired frequency response changes radically across the notch point leading to large interpolation error.

The window method is easy to use and closed form expressions are available for the window coefficients. Several windows have been reported in the literature, such as Hamming, Hann, Blackman, Bartlett, Papoulis, Lanczos, Tukey, Kaiser, Dolph-Chebyshev [26] and Prolate Spheroidal wave sequence [32]. These windows offer various trade-offs between the 3-dB transition bandwidth and stopband attenuation. However, "FIR filters based on the window approach do not yield designs which are optimal in any sense, even if the window is optimal in some sense" [36, p-53].

Vaidyanathan and Nguyen [35] introduced FIR eigenfilters which are optimal in the least squares sense. Here, the objective function is defined only as a sum of the passband and stopband errors; the error of approximation in the transition band is not included. One of the advantages of eigen filters over other FIR filters is that they can be designed to incorporate a wide variety of time domain constraints such as the step response and Nyquist constraint[1] in addition to the usual frequency domain characteristics. This method has also been extended to include flatness constraints in the passband.

Out of all the FIR designs, Parks-McClellan iterative design [25] yields the best results, although, it too has some inherent limitations. Equiripple designs only consider the specified passbands and stopbands but the transition bands are not considered in the numerical solution. In fact, transition regions are considered as '*don't care*' regions in the design procedure.

As a result, the numerical solution may fail, especially in the transition region and for notch filters in particular. For the optimum design, the filter

[1] A Nyquist filter must satisfy the condition: $\omega_p + \omega_s = 2\pi/k$, where ω_p and ω_s are, respectively, the passband and stopband edge frequencies and k is the intersymbol time duration. Such filters are extensively used in digital modem systems and also in multirate signal processing [35].

length is determined by the narrower transition band. If the transition band is wide, the algorithm may fail in the transition region resulting in overshoot of the frequency response [6].

Tian-Hu Yu et al. [37] have proposed two methods for designing the notch filters by exploiting the aforementioned design techniques. In one of the methods, a notch filter ($H(\omega)$) is derived from a lowpass filter ($H_{LP}(\omega)$) by using the relation

$$H(\omega) = 2H_{LP}(\omega) - 1. \tag{2}$$

This transformation provides a change of phase by 180 degrees at the notch frequency ω_d i.e. the designed filter response is of the type NF2 (see Fig. 2). The frequency response $H(\omega)$ may further be sharpened by using the amplitude change function (ACF) [16]. An alternative method in [37] is based on complementing a narrow passband (tone) filter, $B(\omega)$, to obtain the desired notch filter by using

$$H(\omega) = 1 - B(\omega). \tag{3}$$

Obviously, a narrow-band filter $B(\omega)$ will have a large filter order. A number of techniques are, however, available in [1], [4] and [22] for reducing the number of multiplications.

Another method for designing an FIR notch filter was proposed by Er [7] where the symmetry constraint for the coefficients was relaxed and, therefore, the design yields non-linear phase FIR filters. Two procedures have been proposed in [7] for varying the null width. In the first approach, the mean squared error between the desired unity response and the response of the filter over the frequency band of interest is minimized subject to a null constraint and its zero derivative constraint at the frequency of interest. The null width can be increased in discrete steps by setting more derivatives to zero at the notch frequency.

In the second approach, a null power constraint over a frequency band of interest is introduced. This approach is found to be more effective in controlling the null width as compared to the derivative constraint methodology. Both of these approaches adopt optimization techniques which have been efficiently solved in [7]. The limitation of such a design, however, is that it yields non-linear phase and does not provide closed form mathematical formula for computation of design weights.

FIR filters find extensive use where frequency dispersion due to non-linear phase is undesirable, such as in speech processing, digital communication, image processing, etc.. This is the precise reason that a large number of commercial chips carry out signal processing with FIR filters. We, in this paper, discuss some recent design techniques proposed by us, highlighting analytic designs with recursive as well as explicit mathematical formulas for computation of the weights required in the design of FIR notch filters. Also, we have considered notch filters having maximal flatness in the two passbands, i.e. at $\omega = 0$ as well as at $\omega = \pi$ (in the Butterworth sense). We focus our attention on the design of digital notch filters of type NF1 as well as NF2. Linear phase and also nonlinear phase designs have been investigated.

The transfer function $H(z)$ of a causal FIR filter of length N in terms of its unit sample response $h(i)$ is given by

$$H(z) = \sum_{i=0}^{N-1} h(i) z^{-i}. \tag{4}$$

For the linear phase requirement, $h(i)$ must satisfy the constraint:

$$h(i) = \pm h(N-1-i), \quad i = 0, 1, 2, \dots, N-1 \tag{5}$$

The filter length N can be odd or an even integer. However, we choose N to be an odd integer only so as to avoid problems due to fractional delays. By using (4) and (5), the transfer function of a symmetric linear phase FIR filter can be written as [24]

$$H(z) = z^{-n} \sum_{i=0}^{n} d_i \frac{z^i + z^{-i}}{2}, \quad n = \frac{N-1}{2} \tag{6}$$

The weights d_i are related to the unit sample response $h(i)$ by [24]

$$d_i = \begin{cases} h(n), & i = 0 \\ 2h(n-i), & i = 1, 2, \dots, n \end{cases} \tag{7}$$

The frequency response $H(\exp(j\omega))$ of the causal filter may be written as

$$H(z)\Big|_{z=e^{j\omega}} = e^{-j\omega n} H_0(\omega), \tag{8}$$

where $H_0(\omega)$ is the amplitude function[2] given by

$$H_0(\omega) = \sum_{i=0}^{n} d_i \cos(i\omega). \tag{9}$$

[2]In the literature amplitude function $H_0(\omega)$ is also referred to as the "pseudo magnitude function" or "zero phase amplitude response".

For the design of NF2 type filters, we impose the following optimality criteria:

$$H_0(\omega)\big|_{\omega=0} = 1, \tag{10a}$$

$$H_0(\omega)\big|_{\omega=\pi} = -1, \tag{10b}$$

$$\vdots$$

$$\frac{d^u H_0(\omega)}{d\omega^u}\Big|_{\omega=\pi} = 0, \quad u = 1, 2, \dots, 2m-1 \tag{10c}$$

$$\frac{d^\nu H_0(\omega)}{d\omega^\nu}\Big|_{\omega=0} = 0, \quad \nu = 1, 2, \dots, 2(n-m)+1 \tag{10d}$$

Here, m is an integer specifying the degree of flatness at $\omega = \pi$, which can have values within the range $1 \leq m \leq n$. Equations (10) give us $(n+1)$ non-trivial equations which can be solved to compute the weights (d_i's). Filters thus designed are specifically useful in applications where linearity of phase is not an essential requirement. One typical application for such filters is in one-dimensional QMF banks [36].

Linear phase maximally flat notch filter designs of the type NF1 have also been accomplished by using Bernstein polynomials as well as by using notch filter to lowpass filter transformation. These approaches lead to explicit formulas for computation of design weights, as will be shown in the next section.

2 Designs

2.1 Design of linear phase notch filter: Analytical approach [30]

In the methods presented in [7], [37], the weights required for the filter structure are found by using computer-aided optimization techniques. However, by an analytical formulation of this problem, it is possible to find the exact mathematical formula for the weights. We choose the optimality criteria of maximal flatness of the amplitude response (in the Butterworth sense) at $\omega = 0$ and $\omega = \pi$ as given by (10). Such a choice leads us to exact mathematical formulas for computing the design weights. It has been shown in [30] that through this methodology, it is possible to realize, exactly, the desired notch frequency, ω_d, besides meeting the specified rejection bandwidth $\overline{BW}$.

2.1.1 Design

Let the frequency response, $H_d(\omega)$, of the desired FIR digital notch filter be given by

$$H_d(\omega) = \sum_{i=0}^{N-1} h(i)e^{j\omega i}, \tag{11}$$

where $h(i)$ is the impulse response and N is the filter length, the filter order being $N-1$. Imposing symmetry condition, we have [24]

$$h(i) = \pm h(N-1-i). \tag{12}$$

The design requirement is to have non-zero $H_d(\omega)$ both at $\omega = 0$ and $\omega = \pi$. Hence, we take the positive sign in (12) so as to obtain a cosine series. Imposing (12) on (11) and keeping N odd, we obtain

$$h_d(\omega) = e^{-j\frac{\omega(N-1)}{2}} \sum_{i=0}^{\frac{N-1}{2}} D_i \cos(i\omega), \tag{13a}$$

where

$$D_0 = h(\frac{N-1}{2}), \tag{13b}$$

$$D_i = 2h(\frac{N-1}{2} - i), \quad i = 1, 2, \ldots, \frac{N-1}{2} \tag{13c}$$

We derive the desired $H_d(\omega)$ through the use of two maximally flat notch filters belonging to the class $H_m(\omega)$, each of order $N-1$, such that:

(i) $H_m(\omega)$ has m degrees of flatness at $\omega = \pi$ where m can assume $(N-1)/2$ different integer values;

(ii) the notch frequency of $H_m(\omega)$ is ω_m i.e. $H_m(\omega_m) = 0$; and

(iii) $H_m(\omega)$ is positive for $0 \leq \omega < \omega_m$ and negative for $\omega_m < \omega \leq \pi$.

A typical amplitude response of the notch filter $H_m(\omega)$ satisfying the above constraints is shown in Fig. 3 and has 180° phase shift at the notch frequency, ω_m. The reason for taking such a response is to evade discontinuities of $H_m(\omega)$ at $\omega = \omega_m$; also it yields exact formulas for computation of the weights required for $H_m(\omega)$. As is clear from Fig. 3, the magnitude response $|H_m(\omega)|$ is that of a typical notch filter. We choose m in such a way that the resulting amplitude response $H_m(\omega)$ has a zero at $\omega = \omega_m$, where ω_m is just short of ω_d. Let

$$H_m(\omega) = \sum_{i=0}^{n} d_i \cos(i\omega), \quad n = \frac{N-1}{2} \tag{14}$$

where d_i's are the weights to be computed. Such an expression for $H_m(\omega)$ obviously represents amplitude response of an FIR, linear phase digital filter [24]. For $H_m(\omega)$, we impose the optimality criteria as enunciated in (10a) to (10d) (after replacing $H_0(\omega)$ by $H_m(\omega)$, of course).

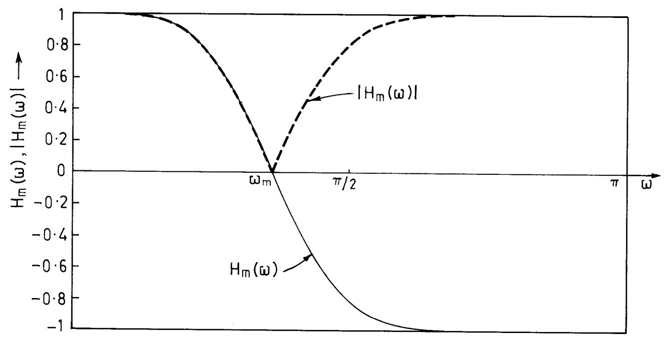

Fig. 3.Frequency response, $H_m(\omega)$ of notch filter with 180° phase shift at the notch frequency, ω_m. The dotted curve gives the magnitude response $|H_m(\omega)|$.

Note that the sum of the possible degrees of flatness at $\omega = 0$ and $\omega = \pi$ is $2n = N - 1$. We need $(n + 1)$ non-trivial equations to solve for the same number of unknown weights d_i's.

The integer m has the range: $1 \leq m \leq n$ giving n different amplitude responses $H_m(\omega)$ for a given value of n. Accordingly, the notch frequency ω_m can assume n discrete values: $\omega_1, \omega_2, \ldots, \omega_n$. Equations (10a) to (10d) give us $(n + 1)$ non-trivial equations. These equations can be put in the matrix form

$$[a_{ij}][d_i] = [b_i]. \tag{15}$$

By using Crout's method [9], and following somewhat involved algebraic manipulations (as in [30]), (15) is transformed to a triangular matrix. The values of d_i's are therefore computed from the recursive formula [30]

$$d_i = b'_i - \sum_{j=i+1}^{n} a'_{ij} d_j \qquad \begin{array}{l} i = n, n-1, \ldots, 0 \\ \text{(descending order)} \end{array} \tag{16}$$

where the values of b'_i and a'_{ij} are given by exact mathematical formulas [30].

Table 1 of [30] gives the values of the weights, d_i's, computed by using (16) for $n = 15$ and m varying from 3 to 15. Knowing the weights, the response curves, $|H_m(\omega)|$ for m varying from 1 to n can be obtained. Figure 4 shows the magnitude response curves for $n = 9$ and $m = 1$ to 9.

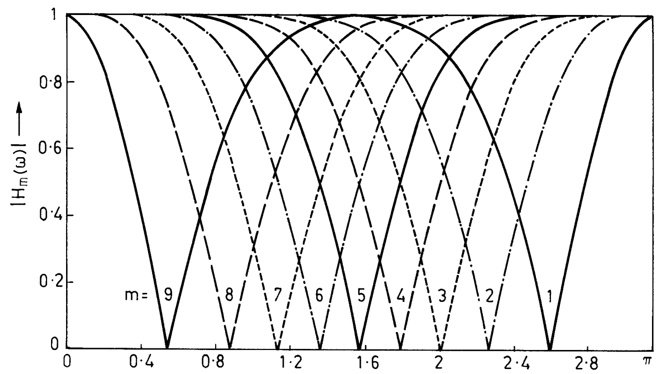

Fig. 4. Magnitude responses, $|H_m(\omega)|$, for linear phase notch filters for $n = 9$ and $m = 1$ to 9.

It is observed that the 3-dB $\overline{BW}$ of $H_m(\omega)$ varies with m (keeping n constant). Also, $\overline{BW}$ progressively decreases as n increases. It is noted that the value of $\overline{BW}$ is maximum for $m = \lfloor (n+1)/2 \rfloor$[3]. Moreover, $|H_m(\omega)|$ has the same $\overline{BW}$ for $m = m_0$ and $m = n + 1 - m_0$. This property of $|H_m(\omega)|$ facilitates computation of weights corresponding to $m = n + 1 - m_0$ from those corresponding to $m = m_0$ by inspection using the relation

$$d_{i,m_0} = (-1)^{i+1} d_{i,n+1-m_0}. \tag{17}$$

An alternative approach to determine the value of n would be to use a suitable empirical formula. Kaiser and Reed [17] have proposed empirical formula for computing the value of n required for a desired transition bandwidth, $\Delta\omega$, of maximally flat lowpass filters.

Empirical formulas suitable for the notch filters are:

$$n \geq \text{Integer}\left\{\frac{1}{2}\left[\left(\frac{\pi}{\overline{BW}}\right)^2 - \frac{\pi}{\overline{BW}} + 3\right]\right\}, \tag{18}$$

[3] $\lfloor x \rfloor$ denotes the integer part of x.

and

$$m_1 = \text{Integer part of}\{n(0.55 + 0.5\cos\omega_d)\}. \tag{19}$$

Formula (19) has been arrived at after modifying an existing formula, due to Herrmann [11], for maximally flat lowpass filters of order n and 3-dB cutoff frequency, ω_c, viz.

$$m = \text{Integer} \leq n\frac{1+\cos\omega_c}{2}. \tag{20}$$

The formulas given by (18) and (19) hold good for filter lengths up to 79.

We design $H_{m_1}(\omega)$ and $H_{m_2}(\omega)$, where $m_2 = m_1 - 1$, and obviously $\omega_{m_1} < \omega_d < \omega_{m_2}$. To obtain desired notch filter $H_d(\omega)$ with notch at $\omega = \omega_d$, we use linear mixing of $H_{m_1}(\omega)$ and $H_{m_2}(\omega)$, i.e.

$$H_d(\omega) = \alpha H_{m_1}(\omega) + \beta H_{m_2}(\omega), \tag{21a}$$

where

$$\alpha = \frac{\omega_{m_2} - \omega_d}{\omega_{m_2} - \omega_{m_1}}, \tag{21b}$$

and

$$\beta = \frac{\omega_d - \omega_{m_1}}{\omega_{m_2} - \omega_{m_1}}. \tag{21c}$$

The weights of the desired notch filters are given by

$$D_i = \alpha d_i^{(m_1)} + \beta d_i^{(m_2)}. \tag{22}$$

Note that α and β satisfy the condition: $\alpha + \beta = 1$; also to ensure that $H_d(0) = -H_d(\pi) = 1$, we should have

$$\frac{\alpha}{\beta} = \left|\frac{H_{m_2}(\omega_d)}{H_{m_1}(\omega_d)}\right|. \tag{23}$$

The aforementioned procedure indeed yields the notch frequency of $H_d(\omega)$, that is very close to the desired one (ω_d). The design also retains the maximal flatness of the passbands and achieves an exact null at the notch frequency. The mathematical formulas for computing the weights needed constitute an attractive feature of this design.

The rejection bandwidth $\overline{BW}$ can be made small if sufficiently high value of n is chosen. For a given n, this filter provides a fixed range of notch frequencies, varying from $\omega_d|_{m=n}$ to $\omega_d|_{m=1}$.

In this procedure, we need a linear combination of $H_{m_1}(\omega)$ and $H_{m_2}(\omega)$, in order to arrive at the desired response $H_d(\omega)$. The value of $H_d(\omega_d)$ is not exactly zero; hence fine tuning is essential to obtain the final response $H_d(\omega_)$ [30]. In the next design, we suggest a semi-analytic method of notch filter design which eliminates the necessity of linear mixing as well as that of the fine tuning.

2.2 A semi-analytic approach for designing FIR notch filter [14]

In this approach, the desired notch filter can be designed directly, without the need for combination of two filters or the requirement of fine tuning. The design requires less number of weights as compared to the analytic approach, given in subsection 2.1.

Let the amplitude response, $H_d(\omega)$, of a typical linear phase FIR digital notch filter be given by [24]

$$H_d(\omega) = \sum_{i=0}^{\frac{L-1}{2}} D_i \cos(i\omega), \tag{24}$$

where D_i's are the weights[4] to be computed, and L is the length of the filter (assumed to be odd). We let $H_d(\omega)$ satisfy the criteria (10a) to (10d) and also the additional constraint:

$$H_d(\omega_d) = 0. \tag{25}$$

The constraint (25) is taken care of by increasing the filter length to $N = L + 2$. It may be noted that, to retain linear phase property of the notch filter, we have increased the length by 2 to obtain one additional nontrivial equation. By using (25) in (24) and taking $n = (N-1)/2$, we have

$$\sum_{i=0}^{n} D_i \cos(i\omega_d) = 0, \tag{26}$$

[4]Note that the weights D_i's used here are not the same as those found in the design given in subsection 2.1 by linear mixing. The notation (D_i) has been retained for convenience.

or

$$\sum_{i=0}^{n} C_i D_i = 0, \tag{27}$$

where

$$C_i \triangleq \cos(i\omega_d), \quad i = 0, 1, 2, \dots, n \tag{28}$$

Imposing the optimality criteria (10a to 10d) on $H_d(\omega)$ and by using (27), we obtain a set of $(n+1)$ non-trivial equations. These equations are again solved by using Crout's method. Here, the recursive mathematical formulas are obtained after somewhat involved manipulation [14]. The value of n is again found preferably by using the empirical formula (18) and m is found from

$$m = \begin{cases} \lfloor n(0.55 + 0.5\cos\omega_d) \rfloor, & 1 \leq n \leq 20 \\ \lfloor n(0.55 + 0.5\cos\omega_d) - 1 \rfloor, & n > 20 \end{cases} \tag{29}$$

where $\lfloor x \rfloor$ denotes the integer part of x. The values of n and m found by above empirical formulas hold good up to $N = 65$. A design example may be referred to in [14].

The magnitude responses of a few other notch filters designed by using the proposed algorithm are shown in Fig. 5. The specified and the realized values of various parameters for these designs are given in [14]. It is seen that in all these designs, we are able to realize the exact notch frequency (ω_d). Also, the realized $\overline{BW}$ is lower than the specified value. This confirms the efficacy of the suggested methodology.

The proposed design method has been found to give the desired frequency response for filter length N upto 65. Beyond this value of N, it has been observed that a small content of ripple appears in the response. We note that the FIR notch filters can also be designed by McClellan and Parks algorithm to obtain equiripple (i.e. minimax) frequency response. We designate such a design by $H_{EQ}(\omega)$. If we compare the performance of notch filters $H_d(\omega)$ with that of filters $H_{EQ}(\omega)$, we find that the 3-dB bandwidth $(\overline{BW}\ )$ is certainly lower in the case of minimax design, as expected. We also note that:

(i) Equiripple/Minimax design using Remez algorithm (i.e. McClellan and Parks approach) is basically iterative in nature and is non-analytic. As pointed out by Rabiner et al., in [28], "An analytical solution to the optimal filter design problem exists for the case of extra-ripple design with either one passband or one stopband ripple.

... these cases are either very wideband or very narrowband designs, and are not generally of much interest, except for the insights they provide into analytical relations between the various parameters". In comparison, the semi-analytic design is non-iterative.

(ii) A simple, workable empirical relation between N and the rejection bandwidth ($\overline{BW}$) specifically for FIR notch filters, is a useful tool for quick design.

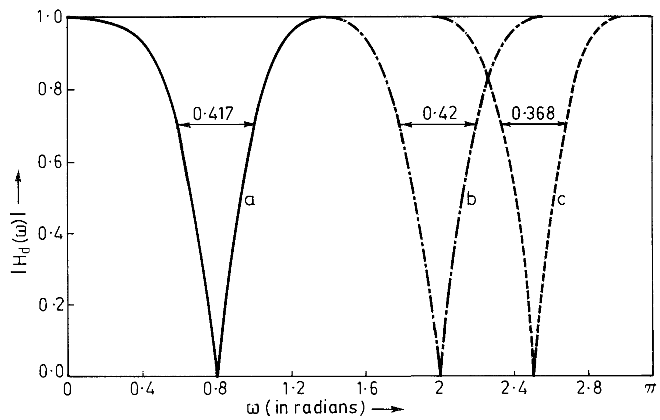

Fig. 5. Ferequency response of notch filters designed for:
(a) ω_d =0.8 radian, 3-dB $\overline{BW}$ =0.44 radian,
(b) ω_d =2.0 radian, 3-dB $\overline{BW}$ =0.42 radian, and
(c) ω_d =2.5 radian, 3-dB $\overline{BW}$ =0.37 radian.

A semi-analytic approach for designing notch filters enables us to realize a notch filter with a specified notch frequency (ω_d) and rejection bandwidth ($\overline{BW}$). The suggested technique has an added advantage that it requires less number of weights than those required by analytical design.

2.3 Design of FIR notch filters from second order IIR prototype [31]

As is well known, IIR filters are highly efficient requiring a much lower order than that needed with the FIR ones. However, IIR filters are potentially unstable due to quantization and limit cycle effects, particularly for highly selective IIR notch filters. In this subsection, we take a typical IIR notch filter of second order as a prototype and evolve a conceptually simpler FIR design to achieve the *same high quality performance without* *any instability problem.* As will be seen, the frequency response of the derived FIR notch filter is indeed very close to that of the prototype IIR notch filter. Mathematical formulas for computing the design weights have also been suggested. These formulas take less time for computing the weights as compared to the designs discussed in subsections 2.1 and 2.2.

2.3.1 The prototype IIR notch filter

IIR digital notch filters can be designed by using classical analog filter approximation methods. However, one is likely to face two types of problems [24] in such designs. The design program requires passband and stopband edge frequencies and ripples as input parameters, and choice of improper specifications can lead to high orders of the filters. Also, for the design of digital narrow-band notch filters, the z-domain poles tend to be very close to the unit circle. This results in a highly non-linear phase response, high round-off noise and potential instability/limit cycles in finite wordlength implementations.

A simpler design strategy proposed by Laakso et al. [19] is to design first and second order IIR prototype notch filters with zeros strictly on the unit circle and poles close to the zeros.

A second order IIR notch filter is, however, more versatile since it can be designed for an arbitrary notch frequency, ω_0. A typical such filter has the transfer function [19]:

$$F_2(z) = K\frac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}. \tag{30}$$

K and r are the parameters as stated earlier. The notch effect in (30) is obtained by placing a pair of complex conjugate zeros at $\exp(\pm j\omega_0)$ while the frequency response in the passband is made close to unity by placing a pair of conjugate poles at $r\exp(\pm j\omega_0)$, where r is less than unity but very close to it. Figure 6 shows the magnitude response of second order IIR notch filter for r varying from 0.9 to 0.99. The 3-dB rejection bandwidth ($\overline{BW}$) of these filters is a function of r and can be reduced by increasing the pole radius r. However, if r is chosen too close to unity, the round-off noise, which is proportional to $1/\Delta^2$ ($\Delta = 1 - r$) [5], becomes very large. For finite wordlengths, these filters also introduce limit cycle problems. This is the precise reason that in many practical applications, we use FIR filters in preference to IIR ones. We, in this design, present two design alternatives for evolving FIR notch filters from the second order IIR prototype given by (30).

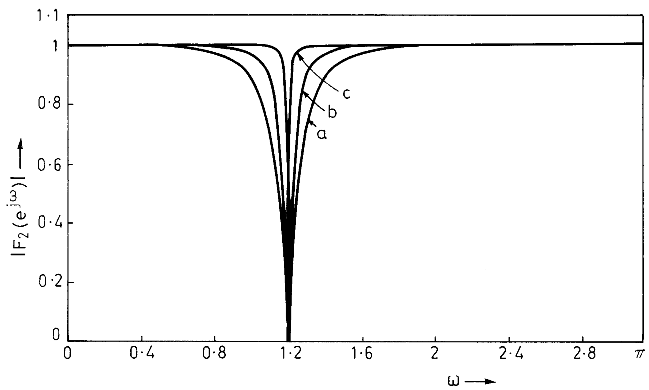

Fig. 6. Magnitude response of second order IIR notch filter [19] having $\omega_0 = 1.2$ rads for $r = 0.90$ (curve a), 0.95 (curve b) and 0.99 (curve c).

2.3.2 FIR design: Approach - I

We rewrite (30) as

$$F_2(z) = KA(z)B(z), \tag{31a}$$

where

$$A(z) = 1 - 2\cos\omega_0 z^{-1} + z^{-2}, \tag{31b}$$

$$B(z) = \frac{1}{1 - az^{-1} + bz^{-2}}, \tag{31c}$$

with

$$a = 2r\cos\omega_0 \quad \text{and} \quad b = r^2. \tag{31d}$$

Clearly,

$$\begin{aligned} A(e^{j\omega}) &= 1 - 2\cos\omega_0 e^{-j\omega} + e^{-j2\omega} \\ &= e^{-j2\omega}(e^{j\omega} - e^{j\omega_0})(e^{j\omega} - e^{-j\omega_0}). \end{aligned} \tag{32}$$

Dividing 1 by $(1 - az^{-1} + bz^{-2})$, $B(z)$ given in (31c) may be expressed as a series with increasing powers of z^{-1}. After simple algebraic manipulations, we get the following elegant form for $B(z)$

$$B(z) = \sum_{i=0}^{\infty} d_i z^{-i}, \tag{33}$$

where

$$d_i = \sum_{m=0}^{\lfloor \frac{i}{2} \rfloor} (-1)^m \binom{i-m}{m} a^{i-2m} b^m, \quad i = 0, 1, 2, \dots , \tag{34}$$

In order to arrive at an FIR structure, we truncate the series for $B(z)$, given in (33), at the $i = M$ term, say, i.e. we approximate $B(z)$ by

$$B_M(z) = \sum_{i=0}^{M} d_i z^{-i}, \tag{35}$$

where the coefficients (d_i's) are functions of r and ω_0 only and are independent of M. This property of d_i's enables us to control $\overline{BW}$ for a given ω_0 and M only by varying r. The resulting FIR filter has the transfer function $H(z) = KA(z)B_M(z)$. Clearly, $H(z)$ is of order $N = M + 2$. The frequency response $H(\exp(j\omega))$ can be readily put in the form

$$H(e^{j\omega}) = 2K(\cos\omega - \cos\omega_0)e^{-j\omega} \sum_{i=0}^{M} d_i e^{-j\omega i}, \tag{36}$$

which is obviously constrained to have a zero (notch) at $\omega = \omega_0$. We may also write

$$H(z) = K \sum_{i=0}^{N} D_i z^{-i}, \tag{37}$$

where

$$D_i = d_i - 2d_{i-1} \cos\omega_0 + d_{i-2}, \tag{38}$$

with $d_k = 0$ for $k < 0$ and $k > M$. The design weight, D_i's, for the proposed FIR notch filter can thus be computed exactly from (34) and (38). We shall investigate the performance of this design after subsection 2.3.3.

2.3.3 FIR design: Approach - II

In the aforementioned treatment, we have truncated the series for $B(z)$ only (keeping $A(z)$ unaltered). Such an approach, obviously, gives an exact zero for $H(\exp(j\omega))$ at $\omega = \omega_0$. Alternatively, we first express $F_2(z)$ as a series of infinite number of terms, and then truncate this series. Clearly, this will not make $H(\exp(j\omega_0))$ exactly equal to zero, but this approach has other advantages over the previous one.

Writing (31a) in the form

$$\begin{aligned} F_2(z) =& K(1 - 2\cos\omega_0 z^{-1} + z^{-2}) \sum_{i=0}^{\infty} d_i z^{-i} \\ =& K \sum_{i=0}^{\infty} \bar{D}_i z^{-i}, \end{aligned} \tag{39}$$

where

$$\bar{D}_i = d_i - 2d_{i-1}\cos\omega_0 + d_{i-2}, \quad i = 0.1, 2, \dots , \tag{40}$$

and $d_k = 0$ for $k < 0$. By using (34) and (40), we arrive at the following explicit formula for $\bar{D}_i$: [31]

$$\bar{D}_i = \sum_{i=0}^{\lfloor \frac{i}{2} - 1 \rfloor} Q(i, m) + C(i), \quad i = 0, 1, 2, \dots , \tag{41a}$$

where

$$\begin{aligned} Q(i, m) =& (-1)^m (2r\cos\omega_0)^{i-2m} r^{2m} \Bigg[\binom{i-m}{m} \\ &+ (2r\cos\omega_0)^{-2} \binom{i-2-m}{m} - r^{-1} \binom{i-1-m}{m} \Bigg], \end{aligned} \tag{41b}$$

and

$$C(i) = \begin{cases} 0, & i = 0, 1 \\ (-1)^{\frac{i}{2}} r^i, & i = 2, 4, 6, \dots , \\ (-1)^{\frac{i-1}{2}} [(i+1)r - 2] r^{i-1} \cos\omega_0, & i = 3, 5, 7, \dots , \end{cases} \tag{41c}$$

An N-th order notch filter, $\bar{H}(z)$, is obtained by truncating the series given by (39) at $i = N$ term, that is

$$\bar{H}(z) = K \sum_{i=0}^{N} \bar{D}_i z^{-i}. \tag{42}$$

The performance of this FIR notch filter $[\bar{H}(z)]$ is given in the next Section, and compared with that of the filters $H(z)$ derived through Approach-I.

2.3.4 Performance of design: - Approaches I & II

The performance of the notch filters designed by using the aforementioned designs (Approaches I and II) has been investigated in respect of

(i) magnitude response,

(ii) relative deviation, and

(iii) group delay

(A) Magnitude response:

The magnitude response $|H(\exp(j\omega)|$ with $M = 50$ (i.e. filter order $N = 52$), $r = 0.85$ and $\omega_0 = 1.2$ rads obtained through (37) (i.e. Approach I) is shown in Fig. 7. For comparison, the magnitude response $|F_2(\exp(j\omega))|$ of the IIR filter designed by using (30) (i.e. the prototype filter) for the same values of r and ω_0 is also shown on the same figure. It is seen that the two magnitude responses viz. $|H(\exp(j\omega))|$ and $|F_2(\exp(j\omega))|$ are indistinguishably close to each other over the entire frequency range $0 \leq \omega \leq \pi$. This indicates that the truncation of the system response $B(z)$ at $M = 50$ has not affected the magnitude response $|F_2(\exp(j\omega))|$ significantly. The 3-dB rejection bandwidth ($\overline{BW}$) in this case (for $N = 52$, $r = 0.85$, $\omega_0 = 1.2$ rads, for example) is found to be 0.3089 rad (= 17.7°). In order to obtain still lower values of $\overline{BW}$, we may increase the value of r. For $r = 0.91$, $N = 52$, and the same notch frequency (i.e. $\omega_0 = 1.2$ rads, for example), the value of $\overline{BW}$ is found to be 0.19 rad (= 10.9°).

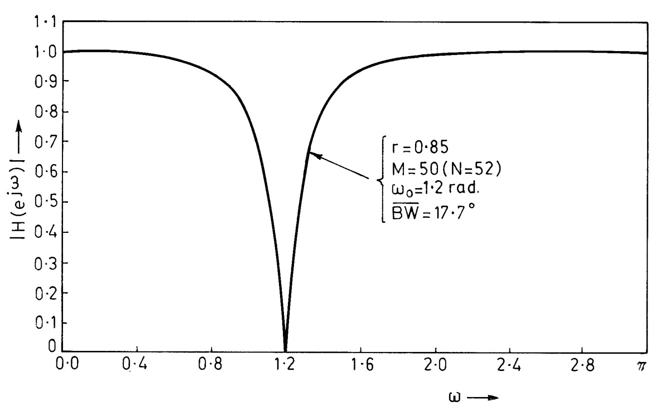

Fig. 7. Magnitude response, $|H(e^{j\omega}|$, (Design Approach-I) for $\omega_0 = 1.2$ rads, $N = 52$ and $r = 0.85$. The magnitude response $|F_2(e^{j\omega})|$ overlaps $|H(e^{j\omega})|$ indistinguishably.

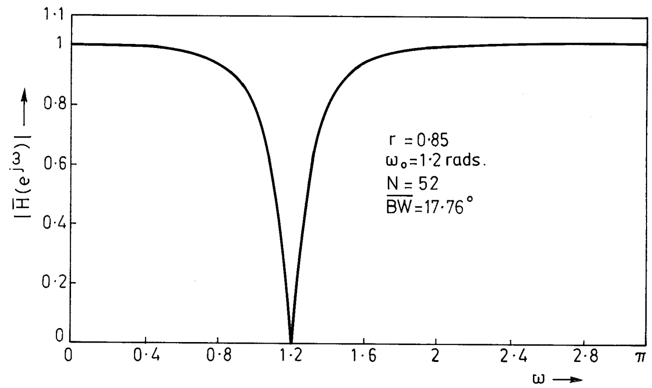

Fig. 8. Magnitude response, $|\bar{H}(e^{j\omega}|$, (Design Approach-II) for $\omega_0 = 1.2$ rads, $N = 52$ and $r = 0.85$.

The magnitude response of the filters designed by using Approach-II is shown in Fig. 8, for $\omega_0 = 1.2$ rads, $N = 52$ and $r = 0.85$. The response obtained here is similar to that shown in Fig. 7. However, if r is changed to 0.91, we obtain the magnitude response, $\bar{H}(\exp(j\omega))$, as shown in Fig. 9.

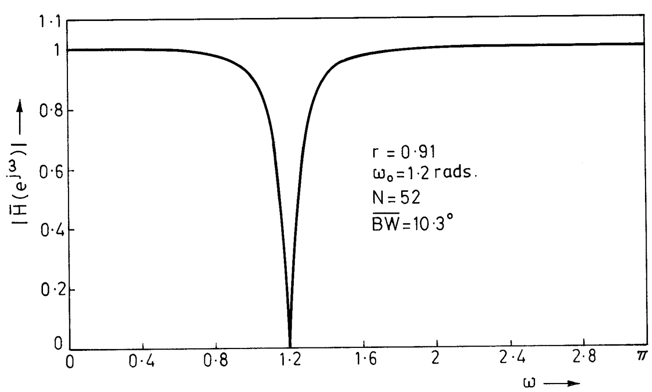

Fig. 9. Magnitude response, $|\bar{H}(e^{j\omega}|$, (Design Approach-II) for $\omega_0 = 1.2$ rads, $N = 52$ and $r = 0.91$.

It is found that this response has relatively less ripple content as compared to that by approach-I (although both the cases here have the same filter order N). Moreover, the $\overline{BW}$, in this case (Fig. 9) is lower ($\overline{BW}$ =10.3°) than that of approach-II (which has $\overline{BW}$ =10.9°). Thus, $\bar{H}(\exp(j\omega))$ results in a closer frequency response to $F_2(\exp(j\omega))$ than that obtained by

$H(\exp(j\omega))$ in the passbands, for the same filter orders of $\bar{H}(z)$ and $H(z)$. We, therefore, infer that the second design alternative can also be used gainfully in applications where the requirement of an exact zero value at the notch frequency is not very stringent, e.g. in adaptive antenna steering or sonar reverberation suppression. The first design approach may be preferred where full cancellation at ω_0 is necessary such as in echo cancellation, or in bio-medical measurements.

(B) Relative deviation:

We define relative deviation for the approximation, say, $G(\exp(j\omega))$ w.r.t the ideal $F_2(\exp(j\omega))$ by

$$E(\omega) \triangleq \left| \frac{|G(e^{j\omega})| - |F_2(e^{j\omega})|}{F_2(e^{j\omega})} \right|, \quad 0 \le \omega \le \pi. \tag{43}$$

Considering the design approach-I, for the case $M = 50$ i.e. $N = 52$, $r = 0.85$, $\omega_0 = 1.2$ rads (Fig. 7) the maximum value, $E_{max}(\omega)$, is found to be 0.000457 i.e. -67 dB. The values of $E_{max}(\omega)$ for $M = 40$ and 45 are found to be -52 dB and -60 dB, respectively, keeping r and ω_0 the same. Thus the relative deviation is reasonably small for the proposed design approach. For the design approach-II, the relative deviation is found to be lower than that of approach-I. For example, for the case of $N = 52$, $r = 0.85$ and $\omega_0 = 1.2$ rads (Fig. 8) the value of $E_{max}(\omega)$ is 0.000169 i.e. -75.4 dB.

(C) Group delay:

It is observed that the group delay responses are indistinguishably close to each other [31]. Thus the conversion of IIR prototype $F_2(z)$ to the proposed FIR designs $H(z)$ as well as $\bar{H}(z)$ does not alter its group delay performance significantly.

In situations where the available memory is rather limited, it would be desirable to have explicit formulas for the weights. The design proposed in subsection 2.3 gives the explicit formula for the weights, but the filter itself has a *non-linear* phase response.

We now give two different design methodologies, by which explicit formulas are obtained for the design of *linear phase* notch filters.

These approaches are based on the use of

(i) Bernstein polynomials, and

(ii) Lowpass filter design.

Both of these designs result in notch filters which are maximally flat at the combination of frequencies $\omega = 0$ and $\omega = \pi$.

2.4 Design of FIR notch filters by using Bernstein polynomials [15]

Here, we use Bernstein polynomials to derive an explicit formula for the weights. This has been possible by expressing the transfer function of the filter as a polynomial in $\cos\omega$. Transfer functions expressed in this form are particularly convenient for implementing variable cutoff filters [23]. We have used Bernstein polynomials because they "mimic the behavior of the function to a remarkable degree" [5, p.116]. Bernstein polynomials are also well known to yield smooth approximations, in contrast to Chebyshev approximations (which is characterized by ripple behaviour). Hence, these polynomials provide an easy method for approximating a function in the maximally flat manner (in the Butterworth sense).

2.4.1 Design

We aim to approximate an ideal notch filter, $H_d(\omega)$ given by

$$H_d(\omega) \triangleq \begin{cases} +1, & |\omega| < \omega_d \\ -1, & \omega_d < |\omega| < \pi \end{cases} \tag{44}$$

by using Bernstein polynomials. Consider a function $f(x)$ defined in the interval [0, 1], as shown in Fig. 10, with functional values given by

$$f(\frac{k}{n}) \triangleq \begin{cases} +1, & 0 \leq k \leq L \\ -1, & L+1 \leq k \leq n \end{cases} \tag{45}$$

where $L+1$ and $n-L$ give the number of successive discrete points at which the function $f(k/n)$ is $+1$ and -1, respectively. The n-th order ($n \geq 1$) Bernstein polynomial for the function $f(x)$ is given by [5]

$$B_n(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}. \tag{46}$$

An alternate expression for (46) is

$$B_n(x) = \sum_{k=0}^{n} \Delta^k f(0) \binom{n}{k} x^k, \tag{47}$$

where $\Delta^k f(0)$ is the k-th forward difference of $f(k/n)$ at $k=0$, and is determined from its functional values at $k = 0, 1, 2, \ldots, n$. From (46), we

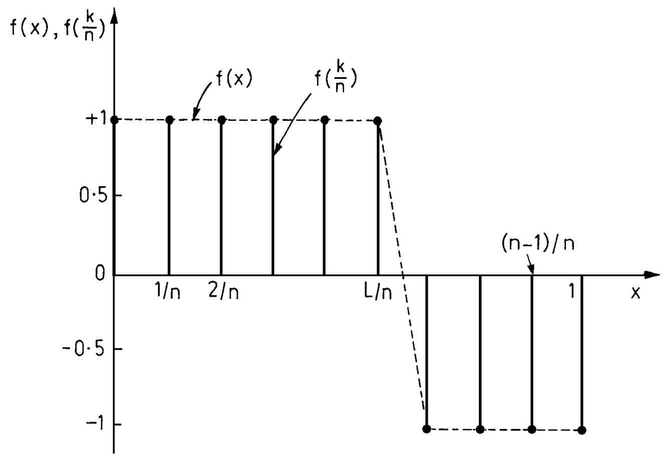

Fig. 10. The function $f(x)$ and $f(k/n)$ used to approximate the notch filters.

note that at extreme points of the range x, the approximating Bernstein polynomial $B_n(x)$ is exactly equal to the value of the desired function $f(x)$ i.e.

$$\begin{aligned} B_n(0) =& f(0) = 1, \\ B_n(1) =& f(1) = -1. \end{aligned}$$

By using $f(k/n)$ as defined in (45) and carrying out some algebraic manipulations, we obtain the following generalized formula for the values of $\Delta^k f(0)$

$$\Delta^k f(0) = \begin{cases} 1, & k = 0 \\ 0, & 1 \le k \le L \\ 2(-1)^{k-L}\binom{k-1}{k-L-1}, & L+1 \le k \le n \end{cases} \tag{48}$$

It is seen from (48) that L forward differences of $f(x)|_{x=k/n}$ are zero at $x = 0$. Therefore, L also signifies the order of flatness of $f(x)$ at $x = 0$ in the Butterworth sense. Using the transformation

$$x = \frac{1 - \cos\omega}{2}, \tag{49}$$

in (47), we have

$$H(\omega) = \sum_{k=0}^{n} \Delta^k f(0) \binom{n}{k} \Big(\frac{1 - \cos\omega}{2}\Big)^k. \tag{50}$$

By using (48), (50) may be written as ($\Delta^k f(0)$ is zero for $1 \le k \le L$) [15]

$$H(\omega) = 1 + \sum_{k=L+1}^{n} 2(-1)^{k-L}\binom{n}{k}\binom{k-1}{k-L-1}\sum_{i=0}^{n} 2^{-k}(-1)^i\binom{k}{i}\cos^i\omega. \tag{51}$$

As L can assume values from 0 to $n-1$, this implies that we can have n different notch filters depending upon the value of L. By simple manipulations, (51) can be reduced to the form

$$H(\omega) = \sum_{i=0}^{n} a_i \cos^i\omega, \tag{52a}$$

where

$$a_i = 2^{-n}\left[2^n\binom{0}{i} + \sum_{k=L+1}^{n}(-1)^{k+i-L}\, 2^{n+1-k}\binom{n}{k}\binom{k-1}{L}\binom{k}{i}\right], \quad i = 0,1,2,\dots,n \tag{52b}$$

In design 2.1, we suggested a methodology for obtaining the desired notch frequency (ω_d) by linear combination of two (out of n) adjacent notch filters mentioned above. In the present context, the procedure gets slightly modified, as given here.

2.4.2 Design procedure and performance

Problem: Given a specified notch frequency ω_d and 3-dB rejection bandwidth ($\overline{BW}$), we are required to design a maximally flat FIR notch filter by using Bernstein polynomial approach.

Step 1: Obtain the required value of n by using the formula

$$n \ge \text{Integer}\left\{\frac{1}{2}\left[\left(\frac{\pi}{\overline{BW}}\right)^2 - \frac{\pi}{\overline{BW}} + 3\right]\right\}. \tag{53}$$

This empirical formula is the same as given in (18).

Step 2: Obtain $L = L_1$ which results in a notch frequency ω_{L_1} closest to but less than ω_d. The value of L_1 is found by using

$$L_1 = (n+1) - \text{Integer part of}\{n(0.55 + 0.5\cos\omega_d)\}. \tag{54}$$

This formula has been obtained by modifying the one given by (19). For $L_2 \triangleq L_1+1$, the corresponding notch frequency, ω_{L_2}, will obviously be closest to but greater than ω_d.

Step 3: The weights of the desired notch filter are obtained by linear mixing of the coefficients $a_i^{(L_1)}$ and $a_i^{(L_2)}$ i.e.

$$a_i = \alpha a_i^{(L_1)} + (1-\alpha) a_i^{(L_2)}, \tag{55a}$$

where

$$\alpha = \frac{\omega_{L_2} - \omega_d}{\omega_{L_2} - \omega_{L_1}}. \tag{55b}$$

A number of notch filters were designed by using the formulas (52b), (53), (54) and (55). Fig. 11, for example, shows the frequency response of a notch filter designed for the specific values: $\omega_d = 1.2$ radians and $\overline{BW}$ $\leq$0.38 radian. The computed values of n, L_1, L_2, ω_{L_1}, ω_{L_2} and α are 31, 10, 11, 1.177783 radians, 1.245955 radians and 0.674, respectively. The realized notch frequency and $\overline{BW}$ are exactly 1.2 radians and 0.38 radian, respectively. Also, the suggested formula (52b) for determining the weights is explicit and requires less memory storage as compared to the recursive formulas proposed earlier.

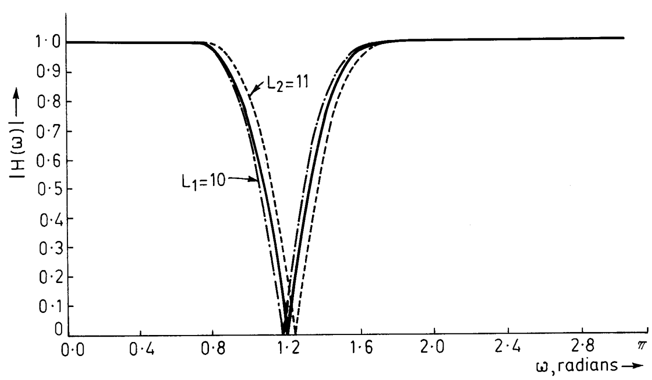

Fig. 11. The frequency response $|H(\omega)|$: for L_1=10 (− · −·), L_2=11 (- - -) and final response (—) for the example considered in Section 2.4.2.

2.5 Design of FIR notch filters by using a lowpass filter [18]

In this design, we present yet another approach for designing linear phase FIR notch filters, which is based on the transformation of a lowpass filter. This method allows realization of the specified notch frequency (ω_d) exactly besides resulting in 3-dB rejection bandwidth ($\overline{BW}$) better than the specified one. The notch filter may be designed with a maximally flat (MF) or equiripple characteristics. We give here the MF design in detail, and then briefly mention the equiripple case.

2.5.1 Design

Let $H_{LP}(\omega)$ be the frequency response of a zero-phase lowpass FIR filter such that

$$\begin{aligned} H_{LP}(0) &= 1, \\ H_{LP}(\omega_d) &= \frac{1}{2}, \quad \text{and} \\ H_{LP}(\pi) &= 0. \end{aligned} \tag{56}$$

We can obtain a notch filter by using the transformation [37]

$$H(\omega) = 2H_{LP}(\omega) - 1. \tag{57}$$

Using (56) in (57) results in the notch filter with

$$\begin{aligned} H(0) &= 1, \\ H(\pi) &= -1, \quad \text{and} \\ H(\omega_d) &= 0. \end{aligned} \tag{58}$$

If $H_{LP}(\omega)$ is maximally flat with flatness distributed between $\omega = 0$ and $\omega = \pi$, then so is $H(\omega)$. The design problem of a notch filter is thus reduced to that of designing a maximally flat lowpass filter $H_{LP}(\omega)$ such that $H_{LP}(\omega) = 1/2$ at $\omega = \omega_d$.

A maximally flat lowpass filter $H_{LP}(\omega)$ can be obtained from Thajchayapong et al. [33] which is a modification of Miller's design [20]. Miller has suggested transforming the zero phase polynomial

$$H_0(z) = \sum_{i=0}^{n} C_i(z^i + z^{-i}), \tag{59}$$

into a rational function $\hat{H}(q)$ ($q = \sigma + j\Omega$) through the bilinear transformation $z^{-1} = (1-q)/(1+q)$. Then by imposing the condition of maximal flatness of $\hat{H}(q)$, for $q = j\Omega$ with m-th degree of flatness at $\Omega = \infty$ (i.e.

$\omega = \pi$, because $\Omega = \tan(\omega/2)$), the resulting analog filter has the transfer function ([33], eqn.2)

$$\hat{H}(j\Omega) = \frac{1}{(1+\Omega^2)^n}\left\{\sum_{k=0}^{n-m}\binom{n}{k}\Omega^{2k}\right\}. \tag{60}$$

Clearly, for a given n, (60) yields n different lowpass filters as m varies from 1 to n. Thajchayapong et al. [33] have suggested a method of obtaining transitional filters between two adjacent values of m, say, m_1 and $m_2 = m_1 - 1$, by modifying (60) as follows

$$\hat{H}_1(j\Omega) = \frac{1}{(1+\Omega^2)^n}\left\{\sum_{k=0}^{n-m_1}\binom{n}{k}\Omega^{2k} + C_{n-m_2}\Omega^{2(n-m_2)}\right\}. \tag{61}$$

The value of constant C_{n-m_2} in (61) is found by forcing $\hat{H}_1(j\Omega)|_{\Omega=\Omega_d} = 1/2$, where $\Omega_d = \tan(\omega_d/2)$. This gives

$$C_{n-m_2} = \frac{1}{\Omega_d^{2(n-m_2)}}\left[\frac{(1+\Omega_d^2)^n}{2} - \sum_{k=0}^{n-m_1}\binom{n}{k}\Omega_d^{2k}\right]. \tag{62}$$

Equation (61) is now transformed back to the z-plane by using the transformation $j\Omega = (1-z^{-1})/(1+z^{-1})$, and we finally obtain [18]

$$\begin{aligned}\bar{H}_1(z) =& 2^{-2n}z^n\left\{\sum_{k=0}^{n-m_1}\binom{n}{k}(-1)^k(1-z^{-1})^{2k}(1+z^{-1})^{2(n-k)}\right.\\ &\left. + C_{n-m_2}(-1)^{n-m_2}(1-z^{-1})^{2(n-m_2)}(1+z^{-1})^{2m_2}\right\}.\end{aligned} \tag{63}$$

By using the Binomial expansion for $(1+z^{-1})^p$, taking causal LPF, $H_2(z) = z^{-n}\bar{H}_1(z)$ and after some manipulations, we finally get [18]

$$\bar{H}_2(z) = \sum_{i=0}^{2n} h(i)z^{-i}, \quad n = \frac{N-1}{2} \tag{64a}$$

where

$$\begin{aligned}h(i) =& 2^{-2n}\left[\sum_{k=0}^{n-m_1}\sum_{q=0}^{2k}(-1)^{k+q}\binom{n}{k}\binom{2k}{q}\binom{2n-2k}{i-q}\right.\\ &\left. + C_{n-m_2}(-1)^{n-m_2}\sum_{q=0}^{2m_2}(-1)^{i-q}\binom{2m_2}{q}\binom{2n-2m_2}{i-q}\right]\end{aligned} \tag{64b}$$

We may also express $h(n-i)$ given by (64) as

$$h(n-i) = \{A(i) + C_{n-m_2}B(i)\}, \tag{65a}$$

where

$$A(i) = 2^{-2n}\sum_{k=0}^{n-m_1}\sum_{q=0}^{2k}(-1)^{k+q}\binom{n}{k}\binom{2k}{q}\binom{2n-2k}{n-i-q}, \tag{65b}$$

$$B(i) = 2^{-2n}\sum_{q=0}^{2m_2}(-1)^{q+m_2+i}\binom{2m_2}{q}\binom{2n-2m_2}{n-i-q}. \tag{65c}$$

$$a_i = \begin{cases} h(n), & i=0 \\ 2h(n-i), & i=1,2,\dots,n \end{cases}. \tag{66}$$

Thus zero phase notch filter is given by

$$H(\omega) = \sum_{i=0}^{n} d_i \cos(i\omega), \tag{67a}$$

where

$$\begin{aligned} d_i &= \begin{cases} 2a_0 - 1, & i=0 \\ 2a_i, & i=1,2,\dots,n \end{cases} \\ &= \begin{cases} 2h(n) - 1, & i=0 \\ 4h(n-i), & i=1,2,\dots,n. \end{cases} \end{aligned} \tag{67b}$$

Hence, the coefficients d_i of the desired notch filter (having notch at $\omega = \omega_d$) can be computed from (67) and (65).

The values of n and m, required to compute $h(n-i)$ are obtained from the empirical formulas (18) and (19). Readers may refer to a design example, given in [18] ,

Equiripple Design:

The aforementioned design procedure can also be adapted for an equiripple notch filter, $\tilde{H}(\omega)$, by using an equiripple LPF $\tilde{H}_{LP}(\omega)$ in (57). If we constrain $\tilde{H}_{LP}(\omega)$ as

$$\tilde{H}_{LP}(\omega) = \begin{cases} \dfrac{2 \pm \delta_1}{2}, & 0 < |\omega| < B_1 \\ \dfrac{1 \pm \delta_2}{2}, & |\omega| = \omega_d \\ \pm\dfrac{\delta_1}{2}, & B_2 < |\omega| < \pi \end{cases}, \tag{68}$$

where $B_2 - B_1 = \overline{BW}$, and δ_1 and δ_2 are, respectively, the maximum ripples in the passband and the stopband of the LPF, then the resulting notch filter will have

$$\tilde{H}(\omega) = \begin{cases} 1 \pm \delta_1, & 0 < |\omega| < B_1 \\ \pm\delta_2, & |\omega| = \omega_d \\ -1 \pm \delta_1, & B_2 < |\omega| < \pi \end{cases} . \tag{69}$$

The equiripple linear phase LPF $\tilde{H}_{LP}(\omega)$ may be obtained by any of the conventional methods such as the McClellan and Parks algorithm [27].

Structure:

Equation (57) suggests that an FIR filter structure meant for lowpass operation can also be used to perform as a notch filter without any additional multiplication (multiplication by 2 amounts to left shift operation). This implies that if we realize an optimal lowpass filter by a linear phase FIR structure, the same can be gainfully exploited as a notch filter without any additional multiplication. The performance of such a notch filter is also optimal.

3. Conclusions

We have given an overview for different design approaches of Notch Filters. In preference to IIR designs, FIR designs are more popular. Several methodologies for designing FIR notch filters have been presented.

We have first proposed, in subsection 2.1, an analytic approach for designing maximally flat, linear phase, notch filters of the type NF2 (i.e. with 180 degrees of phase shift beyond ω_d; Fig.2). Recursive formulas have been derived for computing the coefficients of notch filters in this case. The desired notch filter is obtained by linear combination of two notch filters with notch frequency just below and above the specified notch frequency (ω_d). A '*fine tuning*' is necessary to realize the exact notch frequency. Empirical formulas have been given for finding the values of N (filter length), and m (degree of flatness of amplitude response at $\omega = \pi$) to obtain the desired $\overline{BW}$ and ω_d.

As an improvement over the aforementioned approach, a semi-analytic design has been proposed in subsection 2.2, which does not require linear combination of two filters or even the 'fine tuning'. The computational requirement for obtaining the design weights has been simplified in this approach as compared to that in the analytic approach.

Another straightforward method for designing the notch filters has been proposed in subsection 2.3. Explicit formulas have been derived to determine the weights of maximally flat FIR notch filters by using a second order IIR prototype notch filter. The performance of such a design matches favourably with that of the IIR prototype. However, the filters so designed have non-linear phase response. These designs may be used in some typical applications where linear phase response is not an important consideration.

In yet another approach, in subsection 2.4 , Bernstein Polynomials have been used to obtain an explicit formula for designing linear phase maximally flat FIR notch filters. This design methodology leads to much simpler formulas for computing the weights as compared to the approaches proposed in subsection 2.1 to subsection 2.3. Another design technique has been evolved in subsection 2.5 by transforming the given specifications (of the notch filter) to an equivalent lowpass filter. By exploiting the results proposed in [33] and [37], new explicit formula for the design of linear phase maximally flat notch filters, with exact null at ω_d, have been obtained. Explicit formulas as derived in these approaches have an edge over the recursive formulas from the point of view of computational complexity.

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