FACTA UNIVERSITATIS (NIŠ)
SERIES: ELECTRONICS AND ENERGETICS VOL. 14, NO. 2, AUGUST 2001, 253-270

A NUMERICAL PROCEDURE FOR SOLVING SKIN EFFECT INTEGRAL EQUATION IN THIN STRIP CONDUCTORS

This paper is dedicated to Professor Jovan Surutka on the occasion of his 80th birthday

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Abstract: The new integral equation analysis for skin effect in thin strip conductor is presented. The analytical, iterative and numerical procedure for solving this integral equation is proposed. The numerical procedure is based on the point matching method and polynomial approximation for current density distribution in thin strip conductor. The proposed procedure can be successfully used in skin effect analysis of a large number of strip conductors with various cross sections (flat, circular, elliptical, parabolic, as well as cross section with sharp edges such as L, T, H profiles). The obtained numerical results converge very quickly with increasing of degrees of polynomial current approximation even in the cases when skin effect level is significant. The theoretical investigations are supported by several examples and the obtained results for current density distribution in strip conductors and for resistance and inductance per unit conductors length are presented.

Key words: Skin effect, strip conductors, polynomial approximation, integral equation.

1. Introdution

One of the present authors proposes in Ref.1 a general procedure for obtaining integral equations using differential equations and corresponding boundary conditions. This procedure is used in the present paper for the study of skin effect in thin strip conductors having known, but arbitrary shaped cross section. So one new integral equation for analyzing skin effect

Manuscript received 10 July, 2001.
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is formulated. Unfortunately, with the exception of several geometrically simple cases, such as is the finite-width flat or circular strip conductor, an exact solution of the proposed integral equation can not be obtained analytically. Using this integral equation, an iterative procedure is presented in this paper for the approximate solution of low frequency skin effect [2]. Then the current density distribution in the strip conductor is expanded into infinite decreasing series versus angular frequency. After that, one simple but general numerical procedure for approximate numerical solving of skin effect integral equation is presented [3]. Then the current density distribution in the strip conductor is approximated using finite polynomials with unknown complex coefficients and the point matching method is used for approximate solving of integral equation [4]. Very good convergence and accuracy are obtained, even in the case of high skin effect level. The proposed procedure is very useful for solving skin effect in a large number of strip conductor examples with known, but arbitrary cross sections. Part of the obtained numerical results for current density distribution and for resistance and inductance per unit strip conductor length is presented in the paper.

2. Short Theoretical Presentation

An infinite-length strip conductor of uniform cross section and negligible thickness $b \to 0$ is observed (Fig. 1). Under the assumption that only conductive axial current exists in the conductor and that there is not axial propagation, the function of current density distribution satisfies the following differential equation

$$\frac{\mathrm{d}^2 J}{\mathrm{d}v^2} + k^2 h^2 J = 0, \quad \text{for} \quad u = u_0, v_1 \le v \le v_2, \tag{1}$$

where: $J = J(v)$ denotes the axial current density component in the strip conductor, $k^2 = -j\omega\mu\sigma$, $k = (1-j)/\delta$, f is frequency, $\omega = 2\pi f$ is angular frequency and $\delta = 1/\sqrt{\mu\pi\sigma f}$ is the depth of the penetration of strip material having conductivity σ and magnetic permeability μ. $j = \sqrt{-1}$ is imaginary unit.

The equation (1) has been written in a cylindrical coordinate system u, v, z formed by use of the analytic complex variable function,

$$\underline{w} = f(\underline{z} = x + jy) = u(x, y) + jv(x, y), \tag{4}$$

when the Cauchy-Riemann conditions have been satisfied [5].

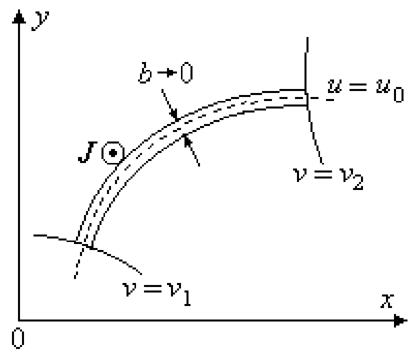


Fig. 1. Cross section of strip conductor.

The strip conductor is put in the coordinate surface $u = u_0$ and extents from $v = v_1$ to $v = v_2$.

Lamè's coefficients are determined by

$$h = h_u = h_v = \left| \frac{\mathrm{d}\underline{z}}{\mathrm{d}\underline{w}} \right|, \quad h_z = 1. \tag{3}$$

After transforming differential equation (1) into form

$$\frac{\mathrm{d}^2 J}{\mathrm{d}v^2} + p^2 J = \left(p^2 - k^2 h^2\right) J, \tag{4}$$

where p is an arbitrary constant, which can be determined at will, the solution of current density distribution can be presented as linear combination of homogeneous and particular integral. So it is possible to formulate the following integral equation for determining current density distribution in a strip conductor,

$$\begin{aligned} J =& C_1 \cos\left[p(v - v_1)\right] + \frac{C_2}{p} \sin\left[p(v - v_1)\right] \\ &+ \frac{1}{p} \int_{v_1}^{v} \left[(p^2 - k^2 h^2) J\right]\Big|_{v=s} \sin\left[p(v - s)\right] \mathrm{d}s, \end{aligned} \tag{5}$$

where C_1 and C_2 are constants.

In the case when $p = 0$, the integral equation becomes considerably simpler and has the following form

$$J(v) = C_1 + C_2(v - v_1) + k^2 \int_{v_1}^{v} (h^2 J)\Big|_{v=s} (s - v)\mathrm{d}s, \tag{6}$$

where $C_1 = J(v = v_1)$ and $C_2 = J^{'}(v = v_1)$

After determining the current density distribution in the conductor, it is possible to calculate the electric and magnetic field components, as well as the resistance and inductance per unit strip conductor length.

So the electric field has in the conductor only axial component,

$$E = \frac{J}{\sigma}. \tag{7}$$

Transversal components of the magnetic filed in the strip conductor are

$$H_u = \frac{1}{k^2 h}\frac{dJ}{dv} \quad \text{and} \quad H_v = 0. \tag{8}$$

The total current in the strip conductor can be determined as

$$\begin{aligned} I =& b\big[H_u(v = v_2) - H_u(v = v_1)\big] \\ \approx& b\int_{v_1}^{v_2} (hJ)\Big|_{u=u_0} \mathrm{d}v. \end{aligned} \tag{9}$$

The power per strip conductor unit length can be expressed as

$$\underline{Z}^{'} II^* = b\left[E(v = v_1)H_u^*(v = v_1) - E(v = v_2)H_u^*(v = v_2)\right], \tag{10}$$

where

$$\underline{Z}^{'} = R^{'} + j\omega L^{'} \tag{11}$$

is impedance per unit conductor length.

$R^{'}$ and $L^{'}$ are resistance and inductance per unit strip conductor length, respectively.

3. About Integral Equation Solving

Unfortunately, with the exception of several geometrically simple cases, such as the finite-width flat (Fig. 2) and circular strip conductor (Fig. 3), an exact solution of the integral equations (5) and (6) can not be obtained analytically.

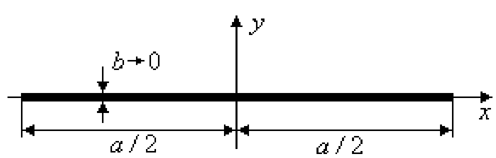


Fig. 2. Thin flat strip conductor.

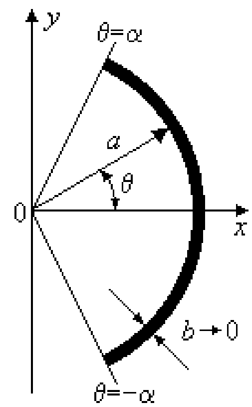


Fig. 3. Thin circular strip conductor.

In the case of flat thin strip conductor two following symmetry conditions exist, $J(-x) = J(x)$ and $J^{'}(x=0) = 0$ and the integral equation (6) can be written as

$$J(x) = J(0) + k^2 \int_{s=0}^{x} J(s)(s-x)\mathrm{d}s, \tag{12}$$

where $J(0)$ denotes the value of current density distribution at strip center point, $x = 0$.

The exact solution for current density distribution is

$$J(x) = J(0)\cos(kx) = J(0)\sqrt{\frac{\mathrm{ch}(2x/\delta) + \cos(2x/\delta)}{2}}\mathrm{e}^{j\,\mathrm{arctg}\left(\frac{\mathrm{th}[x/\delta]}{\mathrm{tg}[x/\delta]}\right)}. \tag{13}$$

The resistance and inductance per unit strip conductor length are

$$R^{'} = \frac{1}{2b\sigma\delta}\frac{\mathrm{sh}(a/\delta) + \sin(a/\delta)}{\mathrm{ch}(a/\delta) - \cos(a/\delta)} \tag{14}$$

and

$$L^{'} = \frac{\mu\delta}{4b}\frac{\mathrm{sh}(a/\delta) - \sin(a/\delta)}{\mathrm{ch}(a/\delta) - \cos(a/\delta)}. \tag{15}$$

For low frequency, when $\delta \to \infty$,

$$R' = R'_0 = \frac{1}{ab\sigma} \text{ and } L' = L'_0 = \frac{\mu a}{12b}. \tag{16}$$

For high frequency, $\delta \to 0$,

$$R' = \frac{R_s}{2b}, \quad L' \to 0, \tag{17}$$

where $R_s = \sqrt{\mu\pi f/\sigma}$ is surface resistance.

In the case of the shell from Fig. 3, the integral equation (6) has the form

$$J(\theta) = J(0) + k^2 a^2 \int_0^{\theta} J(s)(s-\theta)\mathrm{d}s \tag{18}$$

and the exact solution for current density distribution satisfying symmetry conditions $J(-\theta) = J(\theta)$ and $J^{'}(\theta = 0) = 0$ i S

$$J(\theta) = J(0)\cos(ka\theta), \tag{19}$$

where $J(0)$ is the current density in the conductor center point, $\theta = 0$.

The resistance and inductance per unit strip conductor length are

$$R^{'} = \frac{1}{2b\sigma\delta} \frac{\mathrm{sh}(2a\alpha/\delta) + \sin(2a\alpha/\delta)}{\mathrm{ch}(2a\alpha/\delta) - \cos(2a\alpha/\delta)} \tag{20}$$

$$L^{'} = \frac{\mu\delta}{4b} \frac{\mathrm{sh}(2a\alpha/\delta) - \sin(2a\alpha/\delta)}{\mathrm{ch}(2a\alpha/\delta) - \cos(2a\alpha/\delta)}. \tag{21}$$

For low frequency, when $\delta \to \infty$,

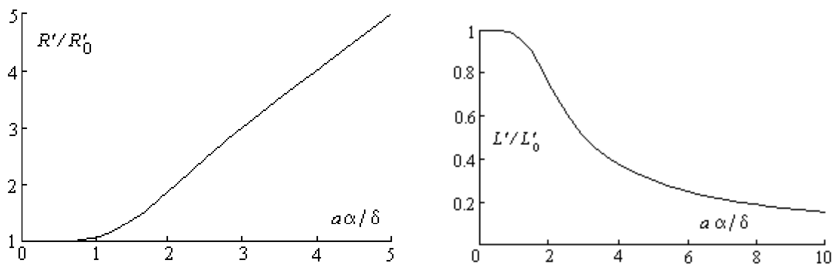
$$R^{'} = R_0^{'} = \frac{1}{2\sigma b a\alpha} \quad \text{and} \quad L^{'} = L_0{'} = \frac{\mu a\alpha}{6b}. \tag{22}$$

For high frequency, $\delta \to 0$,

$$R^{'} = \frac{R_s}{2b} \quad \text{and} \quad L^{'} \to 0, \tag{23}$$

where $R_s = \sqrt{\mu\pi f/\sigma}$ is surface resistance.

In Fig. 4 the resistance and inductance per unit circular strip length versus skin effect level are presented.



(a) (b)

Fig. 4. Resistance (a) and inductance (b) per unit of thin circular strip length versus skin effect level.

4. Iterative Solution of Integral Equation

An approximate solution of the integral equation (6) can be obtained using iterative procedure for low frequency analysis. Then the current density distribution can be expanded in infinite decreasing series versus angular frequency.

By example, for thin strip conductor having parabolic cross section (Fig. 5) the integral equation (6) can be presented as

$$J(u) = C_1 + k^2 \int_{s=0}^{u} (h^2 J)\Big|_{u=s} (s-u)\mathrm{d}s, C_1 = J(0), \tag{24}$$

where the following conformal mapping is used,

$$\underline{w}^2 = j2\underline{z}, \tag{25}$$

so

$$x = uv, \qquad y = \frac{v^2 - u^2}{2} \tag{26}$$

and Lamè's coefficients are determined as

$$h = h_u = h_v = \sqrt{u^2 + v^2}. \tag{27}$$

The parabolic strip conductor is positioned in the coordinate surface $v = v_0$ and extents from $u = -u_0$ to $u = u_0$.

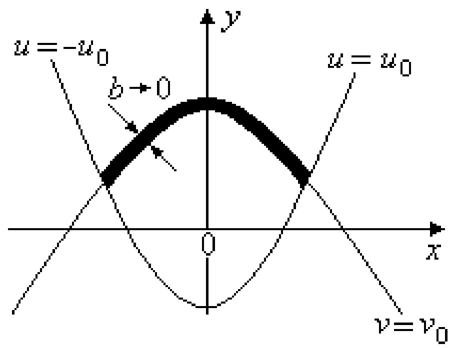


Fig. 5. Thin strip conductor with parabolic form.

The current density distribution can be expanded in infinite series

$$J(u) = \sum_{n=0}^{\infty} J_{2n}(u) k^{2n}, \tag{28}$$

where

$$J_{2n+2}(u) = \int_{s=0}^{u} (s^2 + v_0^2)(s-u) J_{2n}(s) \mathrm{d}s, \quad n = 0, 1, \dots, \quad J_0 = C_1. \tag{29}$$

So it can be obtained:

$$\begin{aligned}
J_2(u) =& -\frac{1}{2}\left(u^2 v_0^2 + \frac{u^4}{6}\right) J_0, \\
J_4(u) =& \frac{1}{24}\left(u^4 v_0^4 + \frac{7}{15} u^6 v_0^2 + \frac{u^8}{28}\right) J_0, \\
J_6(u) =& -\frac{1}{720}\left(u^6 v_0^6 + \frac{11}{14} u^8 v_0^4 + \frac{211}{1260} u^{10} v_0^2 + \frac{5}{616} u^{12}\right) J_0, \\
J_8(u) =& \frac{1}{40320}\left(u^8 v_0^8 + \frac{10}{9} u^{10} v_0^6 + \frac{1201}{2970} u^{12} v_0^4 + \frac{4867}{90090} u^{14} v_0^2 + \frac{u^{16}}{528}\right) J_0, \\
J_{10}(u) =& -\frac{1}{3628800}\Big(u^{10} v_0^{10} + \frac{95}{66} u^{12} v_0^8 + \frac{643}{858} u^{14} v_0^6 + \frac{30973}{180180} u^{16} v_0^4 \\
& + \frac{4031}{2450448} u^{18} v_0^2 + \frac{3}{6688} u^{20}\Big) J_0, \quad \text{etc.}
\end{aligned} \tag{30}$$

5. Numerical Solution of Integral Equation

The integral equations (5) and (6) can be solved numerically in the general case of thin strip conductors of arbitrary form of cross section. Then very good convergence and accuracy can be obtained using point matching method with polynomial approximation for current density distribution,

$$\begin{aligned} J(v) &= \sum_{n=0}^{N} J_n (v - v_1)^n, \\ J_0 &= C_1 = J(0), \\ J_1 &= C_2 = J^{'}(v = v_1), \end{aligned} \tag{31}$$

where J_n are the unknown complex coefficients to be determined.

After the integral equation (6) is satisfied with the solution (31) and the obtained expression left and right sides are matched in as many points, v_{mi}, as the number of unknown complex coefficients, the following linear equations system can be obtained,

$$J(v_{mi}) = J(v_1) + J^{'}(v_1)(v_{mi} - v_1) + k^2 \int_{v_1}^{v_{mi}} (h^2 J)\Big|_{v=s} (s - v_{mi}) ds. \tag{32}$$

Although the matching points can be arbitrarily set, including the conductor's ends, it is natural that they should be selected so as to be equidistant from coordinate v or from the arc length corresponding to the conductor cross section.

After solving linear equations system (32) and determining the unknown complex coefficients, the necessary calculation can be realized on the standard way.

6. Examples

Example 1

Although the exact solution of current density distribution of thin circular strip conductor from Fig. 3 is know (20), the numerical point matching procedure for approximate numerical determining current density distribution in strip conductor will be first presented in this example, in order to verify numerical procedure and compare approximate and exact values.

The current density distribution will be approximated as

$$\frac{J(\theta)}{J(0)} = 1 + \sum_{n=2}^{N} J_n \left|\frac{\theta}{\alpha}\right|^n, \tag{33}$$

so the existing symmetry conditions, $J(\theta) = J(-\theta)$ and $\mathrm{d}J/\mathrm{d}\theta|_{\theta=0} = 0$, are automatically satisfied.

Substituting approximate solution (33) into integral equation (18) and matching the obtained expression the following system of linear equations is obtained

$$\sum_{n=2}^{N} J_n \left[\left(\frac{\theta_i}{\alpha}\right)^n - j\frac{2}{(n+2)(n+1)}\left(\frac{a\alpha}{\delta}\right)^2\left(\frac{\theta_i}{\alpha}\right)^{n+2}\right] = j\left(\frac{a\theta}{\delta}\right)^2, \tag{34}$$

where

$$\theta_i = \alpha\frac{i}{N-1}, \quad i = 1, 2, K, N-1, \tag{35}$$

defines the position of matching points.

Table 1 shows the convergence of the numerically obtained results for the current density distribution, $J(\theta)/J(0)$, when $a\alpha/\delta = 5$ and polynomials of different degrees are used. The exact values calculated by formula (19) are presented in the last column of the same table.

Table 1. Current density distribution in the thin circular strip conductor from Fig. 3, for $a\alpha/\delta = 5$ and different degrees in polynomial current approximation (33).

	Re$\{J(\theta)/J(0)\}$					
θ/α	$N=5$	$N=6$	$N=7$	$N=10$	$N=15$	exact
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.8900198	0.9781713	1.0004733	0.9896027	0.9895849	0.9895849
0.2	0.8435860	0.8037250	0.8315809	0.8337743	0.8337300	0.8337300
0.3	0.4921265	0.1323604	0.1630655	0.1664661	0.1664029	0.1664029
0.4	−1.0630936	−1.5985934	−1.5610738	−1.5655722	−1.5656258	−1.5656258
0.5	−4.5906213	−4.9279098	−4.9316852	−4.9128392	−4.9128446	−4.9128446
0.6	−10.068465	−9.8716399	−10.025937	−9.9670703	−9.9669099	−9.9669098
0.7	−16.023745	−15.167731	−15.571721	−15.520151	−15.519732	−15.519732
0.8	−18.872340	−17.264089	−17.895537	−17.850741	−17.849852	−17.849852
0.9	−12.258539	−9.0500821	−9.7244338	−9.4899778	−9.4887874	−9.4887874
1.0	13.605309	21.6685080	21.087448	21.049511	21.0505562	21.050556

Table 1. Continue.

$\mathrm{Im}\{J(\theta)/J(0)\}$						
θ/α	$N=5$	$N=6$	$N=7$	$N=10$	$N=15$	exact
0.0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.0960814	0.3051422	0.2425678	0.2497954	0.2498264	0.2498264
0.2	0.8080736	0.9819332	0.9993544	0.9889401	0.9888977	0.9888977
0.3	2.1944605	2.0343894	2.1383619	2.1239571	2.1239456	2.1239456
0.4	3.6472770	3.2324871	3.3064377	3.2980262	3.2978948	3.2978948
0.5	4.0209455	3.5825958	3.6393421	3.6210348	3.6208789	3.6208788
0.6	1.7611119	1.2374318	1.4341467	1.4140431	1.4137226	1.4137226
0.7	−4.9665183	−6.1544672	−5.8313155	−5.8026803	−5.8028762	−5.8028762
0.8	−18.147344	−20.659747	−20.747225	−20.652859	−20.653077	−20.653077
0.9	−39.789032	−42.831724	−44.104054	−43.992780	−43.991799	−43.991799
1.0	−71.792682	−70.042347	−71.523628	−71.157009	−71.155260	−71.155260

Table 1. Continue.

$\mid J(\theta)/J(0) \mid$						
θ/α	$N=5$	$N=6$	$N=7$	$N=10$	$N=15$	exact
0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.8951909	1.0246614	1.0246614	1.0206426	1.0206329	1.0206329
0.2	1.1681696	1.2689235	1.2689235	1.2935154	1.2934545	1.2934545
0.3	2.2489654	2.0386907	2.0386907	2.1304705	2.1304541	2.1304541
0.4	3.7990522	3.6061717	3.6061717	3.6507524	3.6506567	3.6506567
0.5	6.1026065	6.0925601	6.0925601	6.1031043	6.1030161	6.1030161
0.6	10.2213262	9.9488950	9.9488950	10.0668768	10.0666729	10.0666729
0.7	16.7757774	16.3687975	16.3687975	16.5694357	16.5691119	16.5691119
0.8	26.1818891	26.9234828	26.9234828	27.2981602	27.2977441	27.2977441
0.9	41.6345877	43.7773984	43.7773984	45.0047147	45.0035050	45.0035048
1	73.0704706	73.3174915	73.3174915	74.2051323	74.2037530	74.2037528

Table 1. Continue.

$\arg\{J(\theta)/J(0)\}$						
θ/α	$N=5$	$N=6$	$N=7$	$N=10$	$N=15$	exact
0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.1075378	0.3023853	0.2378632	0.2472549	0.2472886	0.2472886
0.2	0.7639005	0.8848685	0.8767756	0.8703224	0.8703274	0.8703274
0.3	1.3501877	1.5058264	1.4946864	1.4925807	1.4926099	1.4926099
0.4	1.8544147	2.0300662	2.0119017	2.0139997	2.0140284	2.0140284
0.5	2.4222506	2.5129741	2.5058476	2.5064277	2.5064488	2.5064488
0.6	2.9684308	3.0168909	2.9995128	3.0006617	3.0006910	3.0006910
0.7	3.4421503	3.5270552	3.4999083	3.4993814	3.4994013	3.4994013
0.8	3.9074093	4.0162916	4.0006535	3.9996382	3.9996681	3.9996681
0.9	4.4135283	4.5041579	4.4953728	4.4999279	4.4999491	4.4999491
1	4.8996763	5.0124134	4.9990977	5.0000045	5.0000247	5.0000247

The values of approximate and exact ratio $|J(\theta)/J(0)|$, for various skin effect levels, $a\alpha/\delta = 5$, and 2.5, in thin circular strip conductor from Fig. 3 are presented in the Table 2.

Table 2. Intensity of current density distribution, $|J(\theta)/J(0)|$, in the thin circular strip conductor from Fig. 3, for different ratio $a\alpha/\delta = 5$, 2.5 and 0.5.

	$a\alpha/\delta = 0.5$, $N = 8$		$a\alpha/\delta = 2.5$, $N = 15$		$a\alpha/\delta = 5$, $N = 15$	
θ/α	numerical	exact	numerical	exact	numerical	exact
0.0	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
0.1	1.0000021	1.0000021	1.0013006	1.0013013	1.0206329	1.0206329
0.2	1.0000333	1.0000333	1.0206329	1.0206329	1.2934545	1.2934545
0.3	1.0001687	1.0001687	1.1007146	1.1007150	2.1304541	2.1304541
0.4	1.0005332	1.0005332	1.2934542	1.2934545	3.6506567	3.6506567
0.5	1.0013013	1.0013013	1.6326567	1.6326582	6.1030161	6.1030161
0.6	1.0026966	1.0026966	2.1304544	2.1304541	10.0666729	10.0666729
0.7	1.0049904	1.0049903	2.7961016	2.7961016	16.5691119	16.5691119
0.8	1.0084993	1.0084993	3.6506480	3.6506567	27.2977441	27.2977441
0.9	1.0135818	1.0135819	4.7330578	4.7330394	45.0035050	45.0035048
1.0	1.0206329	1.0206329	6.1030147	6.1030161	74.2037530	74.2037528

Table 3 shows the ratio $R^{'}/R_0^{'}$ and $L^{'}/L_0^{'}$ for different order of power series and different skin effect levels, $a\alpha/\delta$, in thin circular strip conductor from Fig. 3. $R_0^{'}$ and $R_0^{'}$ are static, direct current resistance and inductance per unit conductor length and $R^{'}$ and $L^{'}$ are dynamic values at an angular frequency ω.

Table 3. Resistance and inductance per unit conductor length for thin circular strip conductor from Fig. 3, for different skin effect level and for different degrees in polynomial current approximation.

	$a\alpha/\delta = 0.5$, $N = 8$		$a\alpha/\delta = 2.5$, $N = 15$		$a\alpha/\delta = 5.0$, $N = 15$	
N	$R^{'}/R_0^{'}$	$L^{'}/L_0^{'}$	$R^{'}/R_0^{'}$	$L^{'}/L_0^{'}$	$R^{'}/R_0^{'}$	$L^{'}/L_0^{'}$
3	1.0055429	0.9984305	2.5215024	0.6019019	5.2880075	0.2093928
4	1.0055421	0.9984277	2.4643376	0.6160309	5.1304865	0.2978267
5	1.0055423	0.9984175	2.4736980	0.6111445	4.9484326	0.3110225
6	1.0055424	0.9984167	2.4764487	0.6101848	4.9579549	0.3052080
7	1.0055424	0.9984167	2.4769103	0.6100381	4.9862631	0.3013694
8	1.0055424	0.9984167	2.4769399	0.6100294	4.9973915	0.3001803
9	1.0055424	0.9984167	2.4769371	0.6100302	4.9994541	0.2999849
10	1.0055424	0.9984167	2.4769365	0.6100304	4.9994831	0.2999826
15	1.0055424	0.9984167	2.4769365	0.6100304	4.9993721	0.2999920
exact	1.0055424	0.9984167	2.4769365	0.6100304	4.9993721	0.2999920

As it is seen in the presented tables, very good convergence and accuracy of the numerical results are obtained using low degrees of polynomials in the approximations of current density distribution, even in the case of significant skin effect level.

Example 2

For elliptically shaped thin strip conductor (Fig. 6),

$$u = u_0 = \frac{1}{2} \ln \frac{a+b}{a-b}, \quad -v_0 \leq v \leq v_0,$$

where $\underline{z} = x + jy = c\ \mathrm{ch}\underline{w}$, $\underline{w} = u + jv$, $x = c\ \mathrm{ch}u \cos v$, $y = c\ \mathrm{sh}u \sin v$, $c = \sqrt{a^2 - b^2}$ is the eccentricity and $a = c\ \mathrm{ch}u_0$ and $b = c\ \mathrm{sh}u_0$ are semi-axes, Lamè's coefficients are

$$h = h_u = h_v = c\sqrt{\frac{\mathrm{ch}(2u) - \cos(2v)}{2}}, \tag{36}$$

so the integral equation (6) is

$$J(v) = J(0) + k^2c^2 \int_0^v (\mathrm{sh}^2 u_0 + \sin^2 s) J(s)(s - v)\mathrm{d}s. \tag{37}$$

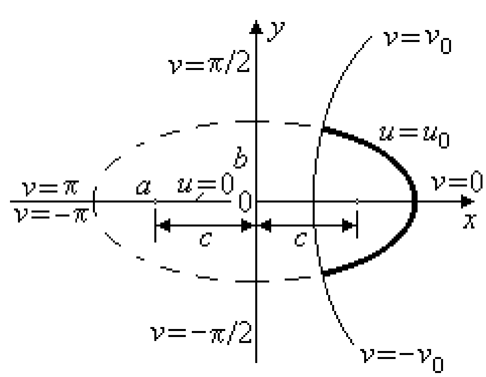


Fig. 6. Thin strip conductor with elliptical cross section.

The integral equation (37) is approximate numerical solved using the presented point matching procedure with the following approximation for current density distribution,

$$\frac{J(v)}{J(0)} = 1 + \sum_{n=2}^{N} J_n \left| \frac{v}{v_0} \right|^n , \tag{38}$$

so the existing symmetry conditions, $J(v) = J(-v)$ and $|\mathrm{d}J/\mathrm{d}v|_{v=0} = 0$, are automatically satisfied.

The modulus and argument of the ratio $J(v)/J(0)$ versus v/v_0 and x/a, for $c/\delta = 0.5$ at $N = 8, c/\delta = 1.5$ at $N = 15$ and $c/\delta = 3$ at $N = 25$ are presented in Figs. 7 and 8, where $a/c = 2b/c = 2/\sqrt{3}$ and $v_0 = \pi/2$.

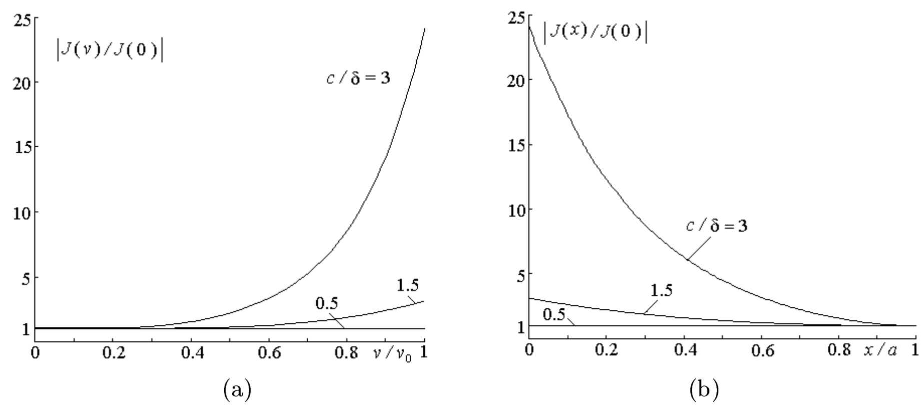


Fig. 7. Intensity of current density distribution in elliptically shaped thin strip conductor versus v/v_0 and x/a, for different ratio c/δ, when $a/c = 2b/c = 2/\sqrt{3}$ and $v_0 = \pi/2$.

The ratios R'/R'_0 and L'/L'_0 for different degrees in polynomial approximation of current density distribution and for different skin effect level are presented in Table 4. R'_0 and L'_0 are static, direct current resistance and inductance per unit conductor length and R' and L' are dynamic values at an angular frequency ω.

Example 3

For V-shaped thin strip conductor (Fig. 9), $u = 0, -\pi \leq v \leq \pi$, where

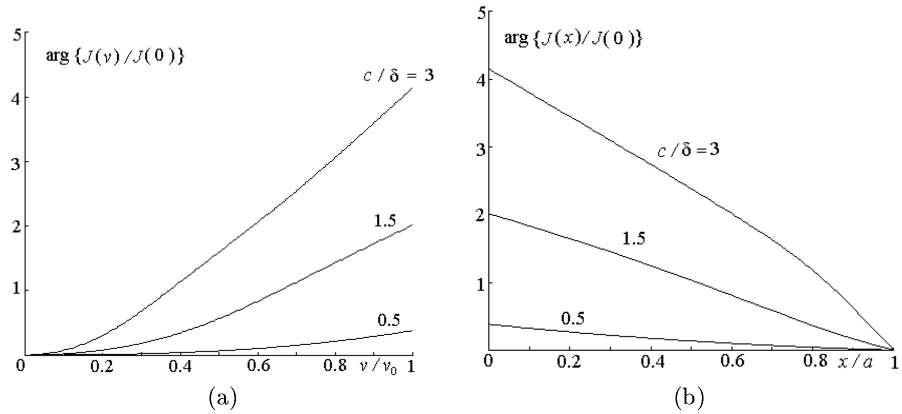


Fig. 8. Argument of current density distribution in elliptically shaped thin strip conductor versus v/v_0 and x/a, for different ratio c/δ, when $a/c = 2b/c = 2/\sqrt{3}$ and $v_0 = \pi/2$.

Table 4. Resistance and inductance per unit conductor length for elliptically shaped thin strip conductor from Fig. 6, for different ratio c/δ, when $a/c = 2b/c = 2/\sqrt{3}$ and $v_0 = \pi/2$.

	$c = 0.5\delta$		$c = 1.5\delta$		$c = 3\delta$	
N	R'/R'_0	L'/L'_0	R'/R'_0	L'/L'_0	R'/R'_0	L'/L'_0
3	1.0133119	0.9929766	1.7353386	0.7361778	3.5500935	0.2454963
4	1.0130925	0.9990073	1.6879252	0.7733736	3.5558465	0.3715012
5	1.0132055	0.9954273	1.6859951	0.7711668	3.4189485	0.4071383
6	1.0132095	0.9953351	1.6892845	0.7683768	3.3927854	0.4060817
7	1.0132079	0.9953726	1.6904633	0.7675407	3.4084515	0.4003042
8	1.0132079	0.9953732	1.6905789	0.7674676	3.4218464	0.3971080
9	1.0132079	0.9953731	1.6905447	0.7674892	3.4265610	0.3961528
10	1.0132079	0.9953732	1.6905304	0.7674977	3.4271683	0.3960421
20	1.0132079	0.9953732	1.6905293	0.7674983	3.4267514	0.3961145
30	1.0132079	0.9953732	1.6905293	0.7674983	3.4267513	0.3961145

$$\underline{z} = x + jy = C\frac{(\mathrm{e}^{\underline{w}} - 1)^p(\mathrm{e}^{\underline{w}} + 1)^{2-p}}{\mathrm{e}^{\underline{w}}}, \underline{w} = u + jv, \tag{39}$$

$$C = \frac{c}{2(2-p)}\left(\frac{2-p}{p}\right)^{p/2}, 0 \le p \le 1, \tag{40}$$

Lamè's coefficients on the strip surface are

$$h = h_u = h_v = 2C\cos|v + p - 1|\left|\mathrm{tg}\frac{v}{2}\right|^{p-1}, \tag{41}$$

so the integral equation (6) is

$$J(v) = J(0) + 4k^2C^2 \int_0^v (\cos s + p - 1)^2 \left(\mathrm{tg}\frac{s}{2}\right)^{2p-2} J(s)(s-v)\mathrm{d}s, \qquad (42)$$

where $J(0)$ denotes the value of current density at the point B_1.

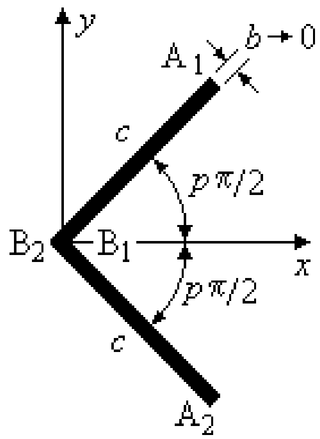


Fig. 9. Thin strip conductor with V-shaped cross section.

The coordinates of characteristic points on V-shaped cross section are given in Table 5.

Table 5. Position of characteristic points of thin strip with V-shaped cross section.

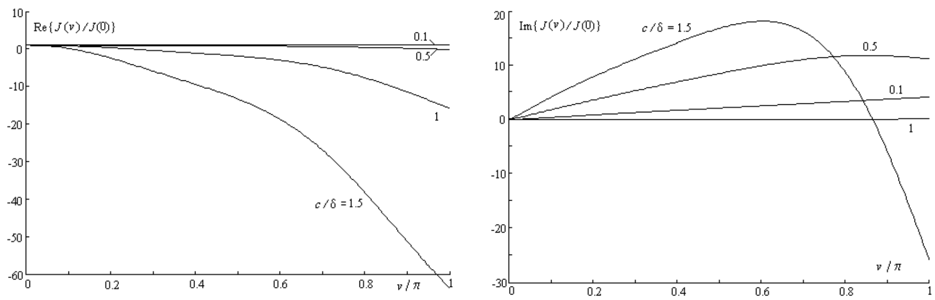
point	x	y	u	v
B_1	$x \to +0$	$y = 0$	$u = 0$	$v = 0$
A_1	$x = c\cos\frac{p\pi}{2}$	$x = c\sin\frac{p\pi}{2}$	$u = 0$	$v = \mathrm{arctg}\frac{\sqrt{2p-p^2}}{1-p}$
B_2	$x \to -0$	$y = 0$	$u = 0$	$v = \pm\pi$
A_2	$x = c\cos\frac{p\pi}{2}$	$x = -c\sin\frac{p\pi}{2}$	$u = 0$	$v = -\mathrm{arctg}\frac{\sqrt{2p-p^2}}{1-p}$

The integral equation (42) is approximate numerical solved using the presented point matching procedure with the following approximation for current density distribution,

$$\frac{J(v)}{J(0)} = 1 + \sum_{n=2}^{N} J_n \left|\frac{v}{\pi}\right|^n, \qquad (43)$$

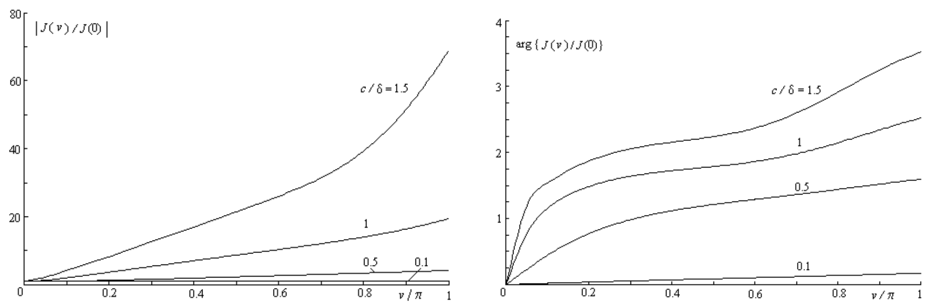
so the existing symmetry conditions, $J(v) = J(-v)$ and $\left.\frac{\mathrm{d}J}{\mathrm{d}v}\right|_{v=0} = 0$, are automatically satisfied.

The real part, imaginary part, modulus and argument of the ratio $J(v)/J(0)$ versus v/π and x/a, for $c/\delta = 0.5$ at $N = 8$, $c/\delta = 1.5$ at $N = 15$ and $c/\delta = 3$ at $N = 25$ are presented in Figs. 10 and 11.



(a) (b)

Fig. 10. Real part (a) and imaginary part (b) of current density distribution in V-shaped thin strip conductor versus v/π for different ratio c/δ.



(a) (b)

Fig. 11. Intensity (a) and argument (b) of curent density distribution in V-shaped thin strip conductor versus v/π for different ratio c/δ.

The ratios $R^{'}/R_0^{'}$ and $R^{'}/R_0^{'}$ for different degrees in polynomial approximation of current density distribution and for different skin effect level are presented in Table 6. $R_0^{'}$ and $L_0^{'}$ are static, direct current resistance and inductance per unit conductor's length and $R^{'}$ and $L^{'}$ are dynamic values at an angular frequency ω.

Table 6. Resistance and inductance per unit conductor length for V-shaped thin strip conductor from Fig. 9, for different ratio c/δ.

c/δ	R'/R'_0	L'/L'_0
0.1	1.002	1.000
0.5	3.100	0.877
1.0	5.969	0.448
1.5	8.038	0.312

6. Conclusion

A new integral equation is proposed for the skin effect solution in thin strip conductors of arbitrary, but known cross section. Several approaches for solving this integral equation are presented: exact analytical, iterative and approximate numerical point matching procedure with polynomial approximation for current density distribution in the strip conductors. The numerous calculations show that the proposed point matching method with polynomial approximation of current density distribution in strip conductor gives very exact results which converge very quickly with increasing of degrees of polynomials in the approximations of current density distribution, even in the case when skin effect level is significant.

Since the presented integral equation is founded on the conformal mapping, it may be concluded that the proposed procedure is applicable in a large number of examples when strip conductors cross sections are very different and complex.

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