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Using negated control signals in Quantum Computing Circuits

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Abstract: White dots have been used in the schematic representation of reversible circuits to indicate that a control variable has to be inverted to become active. The present paper argues that the use of negated control signals may also offer advantages for the realization, by reducing the number of elementary components. In the case of quantum circuits, this contributes to reduce the quantum cost. It is shown that mixed polarity Reed Muller expressions, possibly extended with Boolean disjunctions, are very helpful to design quantum computing circuits including negated control signals.

Keywords: RM expressions; white dots; quantum computing circuits; negated control signals.

1 Introduction

Reversible computing is particularly attractive, because it contributes to reduce power dissipation [1, 2] in digital circuits. It has applications in nanotechnology, low power adiabatic CMOS, and optical computing. Reversible computing is of particular interest for research in quantum computing, since "quantum gates" must be reversible [3].

There is substantial work done on the design of reversible circuits (see e.g. [4–7]). In this paper the method based on Mixed Polarity Reed Muller expressions and hybrid Reed Muller - De Morgan expressions will be specially analyzed, together with controlled gates using negated control signals. The gates set NOT, CNOT, Toffoli will be used in the discussed examples. Toffoli gates are sometimes also called CCNOT gates. "C" stands for "controlled". Whenever control-signals

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have the value 1, the corresponding controlled gate takes an action; otherwise it behaves as the identity. Toffoli-type gates will be introduced, where 0-valued control signals contribute an activation. The symbols used for these gates are shown in Fig. 1, where $\oplus$ denotes an inverter whenever it stands alone or an XOR when it is controlled. (Recall that $x \oplus 1 = \overline{x}$) The black dots denote the conjunction of control variables. Should an action take place when one (or more) control signals have the value 0, up to now, they should first be complemented. Several authors are using "white dots" instead, to indicate that a control variable is effective, when it has the value 0. (See Fig. 2). It is easy to see that a single white dot is equivalent to an inverter, meanwhile two white dots in a Toffoli gate corresponds to a NOR gate.

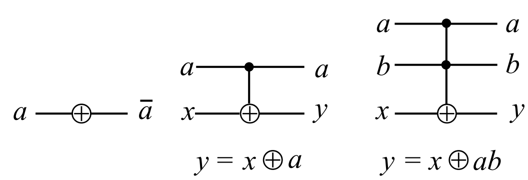


Fig. 1. Inverter and controlled gates.

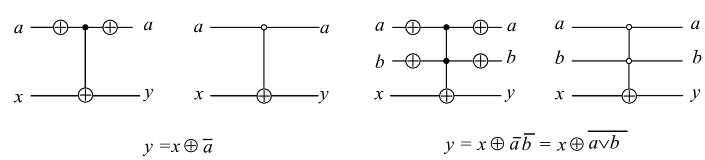


Fig. 2. Negated control signals and equivalent white dot representations.

2 Design based on the Reed Muller transform

The selected set of gates to realize reversible circuits, may be associated to the set $\{\text{AND, XOR, } 1\}$ used for digital circuits. Consequently, methods to obtain AND-XOR expressions (see e.g. [8]), particularly, the well known Reed Muller transform, will be used here [9].

Example 1 Let $f : [0,1]^3 \to [0,1]$ be specified by the truth vector $F = [1,1,0,1,0,0,0,1]^T$ and let the arguments be ordered as x_2, x_1, x_0.

By using the space efficient algorithm introduced in [10], the Reed Muller spectrum of F may be calculated as follows:

$$
\begin{aligned}
S_f =& \boldsymbol{RM}_3 \cdot \boldsymbol{F} = vec(\boldsymbol{RM}_2 \cdot (vec^{-1}(\boldsymbol{F})) \cdot \boldsymbol{RM}_1^T) \\
S_f =& vec\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)
\end{aligned} \tag{1}
$$

where $vec(\boldsymbol{X})$ denotes vectoring the matrix X by ordering its columns to build a vector

$$
S_f = vec \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}
$$

and since the used Reed Muller basis is

$$
\begin{aligned}
B_{RM} &= [1 \quad x_2] \oplus [1 \quad x_2] \oplus [1 \quad x_0] \\
&= [1 \quad x_0 \quad x_1 \quad x_1x_0 \quad x_2 \quad x_2x_0 \quad x_2x_1 \quad x_2x_1x_0]
\end{aligned}
$$

then

$$
f(x) = \boldsymbol{B}_{RM} \cdot \boldsymbol{S}_f = 1 \oplus x_2 \oplus x_1 \oplus x_2x_1 \oplus x_1x_0 \tag{3}
$$

leading to the "one-to-one" naive realization shown in Fig. 3.

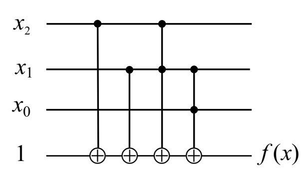


Fig. 3. "Naive" reversible realization of (3).

However it is simple to show that since $x_2 \oplus x_1 \oplus x_2x_1 = x_2 \vee x_1$ and $x \oplus 1 = \overline{x}$, then Eq. (3) turns into

$$
f(x) = \overline{x_2 \vee x_1} \oplus x_1x_2 = \overline{x}_2\overline{x}_1 \oplus x_1x_2 \tag{4}
$$

where the first expression represents a "hybrid Reed Muller DeMorgan" expression, combining GF(2) and Boolean operations. Equation (4) leads to the realizations shown in Fig. 4.

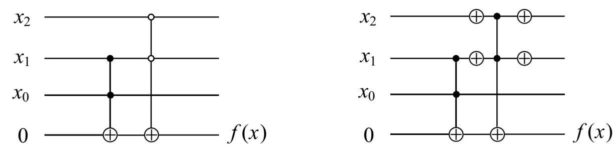


Fig. 4. Reversible realizations of Eq.(4).

Remark: In this simple example, a direct inspection of Eq. (3) allowed to obtain Eq. (4). For more complex cases this complexity reduction would be very difficult, if at all possible. However, in the context of Reed Muller expressions, Eq. (4) represents a (possibly optimal) *mixed polarity* version of Eq. (3). Finding the optimal mixed polarity for a given Reed Muller expression is in *NP*, but there are some good heuristics in the literature (see e.g. [8, 11, 12]). Several of these heuristics start from a fixed polarity expression. The better the fixed the polarity initial expression, the better the expected mixed polarity resulting expression. Finding the optimal fixed polarity expression is also in NP, but the algorithm disclosed in [13] to obtain the optimal polarity is possibly the fastest known.

For instance, the optimal fixed polarity expression for Eq. (3) is:

$$f(x) = \bar{x}_2\bar{x}_1 \oplus \bar{x}_1 x_0 \oplus x_0 \tag{5}$$

which obviously suggests reducing $\bar{x}_1 x_0 \oplus x_0$ to $x_0(\bar{x}_1 \oplus 1) = x_1 x_0$.

The key question is however, whether a "NOR controlled Toffoli" gate, as required to realize Eq. (4) has a simple quantum realization without using additional inverters for the control signals? And in the most general case, are there quantum efficient realizations for generalized Toffoli gates of type $T(x,y,z) = (x,y,z \oplus f(x,y))$, where f denotes any Boolean function?

Notice that Eq. (4) suggests the pseudo-equivalence *claimed* in Fig. 5, where at the right hand side, black inverted triangles are used to represent the OR of control signals, considering their similarity with the $\vee$ sign used in logic to denote a disjunction.

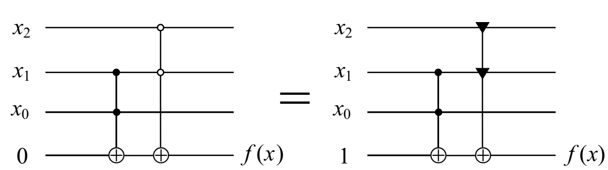


Fig. 5. Hypothetical equivalent realizations.

3 Generalized Toffoli gates

3.1 The OR-Toffoli gate

It is simple to deduce the unitary matrix representation of an OR(CC)NOT gate, as shown below. Important is however, whether an efficient quantum realization for it is possible.

$$OR(CC)NOT \leftrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Following [3] and taking in account that

$$x \vee y = \frac{1}{2}[x + y + (x \oplus y)] \tag{6}$$

where "+" denotes the arithmetic sum, a simple quantum realization with a cost of 5 is possible, as shown in Fig. 6, where **V** denotes the "square root of NOT". The CNOT gate surrounded by a dotted line does not affect z'; it only recovers the input value of y as control variable when $x = 1$.

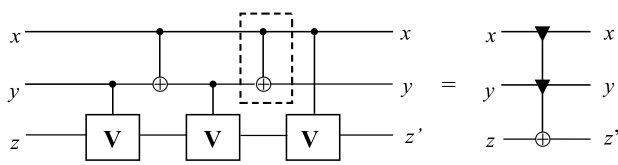


Fig. 6. Quantum realization and symbolic representation of an OR-Toffoli gate.

It is fairly simple to see that if z is free to be an ancillary qubit, then with $z = 0$, $\mathrm{OR}(x,y)$ is obtained, meanwhile with $z = 1$, $\mathrm{NOR}(x,y)$ is obtained. By the way, this NOR-gate would provide an efficient realization of Eq. (4) and supports the claim illustrated in Fig. 5.

3.2 Toffoli gates with one complemented input

A simple analysis of the structure of the efficient realization of a Toffoli gate disclosed in [3] allows obtaining controlling states $|01\rangle$ and $|10\rangle$ *without* needing an additional inverter at the respective input.

Consider the structure shown in Fig. 7, where **X**, **Y** and **Z** denote arbitrary unitary matrices. The gate surrounded by dash-lines satisfies reversibility by recovering the y control signal when $x = 1$. Table 1 summarizes the behaviour of the circuit under different control inputs.

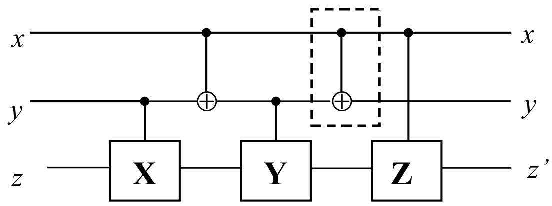


Fig. 7. Abstract "Barenco-type" structure.

Table 2 shows the resulting extensions on Toffoli gates with a quantum cost of 5. It is fair to mention that one of these "new" gates was mentioned in [14], but however does not seem to have received much attention in the RC/QC community. Furthermore notice that if in Fig. 7 the gate surrounded by dash-lines is deleted, a generalization of the Peres gates is obtained. The

Table 1. Generalized Toffoli gates

Behaviour of the generalized gates					
$\|x\rangle$	$\|y\rangle$	**X**	**X**	**X**	$\|z'\rangle$
$\|0\rangle$	$\|0\rangle$	no	no	no	$\|z\rangle$
$\|0\rangle$	$\|1\rangle$	yes	yes	no	$\mathbf{XY}\|z\rangle$
$\|1\rangle$	$\|0\rangle$	yes	yes	yes	$\mathbf{YZ}\|z\rangle$
$\|1\rangle$	$\|1\rangle$	yes	no	yes	$\mathbf{XZ}\|z\rangle$

4 Illustrative cases

Black and white dots may be arbitrarily combined to generate control structures with more than two control signals, taking in account that white dots "react" to 0-valued control signals without inverting them. In Fig. 8, for instance, the first two circuits have a direct realization with gates of Table 2.

In the circuit at the right of Fig. 8, some form of cascading gates with two control signals and an ancillary line may be considered. The expected output is:

$$z'' = z' \oplus \overline{a}\overline{b}c = z' \oplus \overline{(a \vee b)}\, c \qquad (7)$$

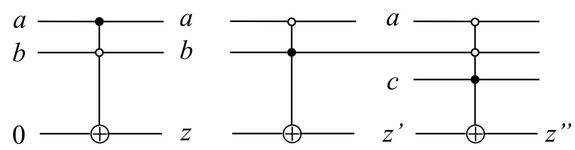
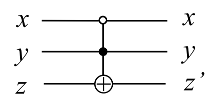
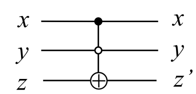
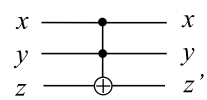


Fig. 8. Mixed control signals.

Table 2. Toffoli gates accepting also negated control signals

By choosing $\mathbf{X} = \mathbf{Y} = \mathbf{Z}$ and $\mathbf{Z} = \mathbf{V}^a$ $z' = z \oplus \text{AND}(\text{NOT}(x), y)$	
By choosing $\mathbf{Y} = \mathbf{Z} = \mathbf{V}$ and $\mathbf{X} = \mathbf{V}^a$ $z' = z \oplus \text{AND}(x, \text{NOT}(y))$ [9]	
By choosing $\mathbf{X} = \mathbf{Z} = \mathbf{V}$ and $\mathbf{Y} = \mathbf{V}^a$ $z' = z \oplus \text{AND}(x, y)$ [3]	

It is simple to see that an AND-NOR-Toffoli decomposition is fairly obvious. Fig. 9 shows an efficient realization, where by using Peres-based OR gates a total quantum cost of 13 may be achieved. Notice that for reversibility, the second OR-(Peres) gate recovers b and generates $a \vee (a \oplus b)$, which equals $a \vee b$. Therefore, the 1 ancillary qubit is also recovered.

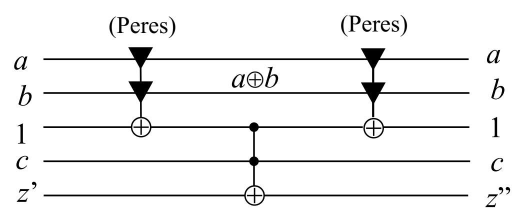


Fig. 9. Realization of $z'' = z' \oplus \bar{a}\bar{b}c = z' \oplus \overline{(a \vee b)}c$ with a quantum cost of 13.

Notice that by adapting the method recently disclosed in [15] or by extending the analysis shown in Fig. 7 and Table 1 to the scheme shown in section 7 of [3], as discussed in [16], a realization with also a quantum cost of 13, *but without requiring an ancillary line* is obtained, as shown in Fig. 10. In this case, elementary unitary matrices W and W^* are however required, where W^4 =NOT and W^* denotes the complex conjugate (and inverse) of W.

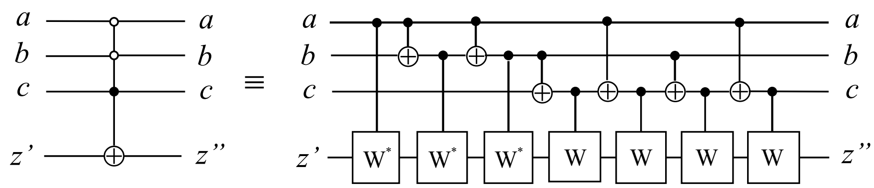


Fig. 10. Realization of $z'' = z' \oplus \bar{a}\bar{b}c = z' \oplus \overline{(a \vee b)}c$ with a quantum cost of 13 and no ancillary line.

Example 2 Example 2: Let $f : \{0,1\}^4 \to \{0,1\}$ be specified by the truth vector $\mathbf{F} = [1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0]^T$, and let the arguments be ordered as x_3, x_2, x_1, x_0. Its Reed Muller Spectrum is given by

$$\mathbf{S}_f = \mathbf{RM}_4 \cdot \mathbf{F} = vec(\mathbf{RM}_2 \cdot (vec^{-1}(\mathbf{F}) \cdot \mathbf{RM}_2^T) \tag{8}$$

$$\begin{aligned}
\mathbf{S}_f &= vec\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \\
\mathbf{S}_f &= vec\left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = vec \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
\mathbf{S}_f &= [1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0]^T
\end{aligned} \tag{9}$$

Since the used Reed Muller basis is

$$\begin{aligned}
\mathbf{B}_{RM} =& [1 \quad x_3] \oplus [1 \quad x_2] \oplus [1 \quad x_1] \oplus [1\ x_0] \\
=& [1 \quad x_0 \quad x_1 \quad x_1x_0 \quad x_2 \quad x_2x_0 \quad x_2x_1 \quad x_2x_1x_0 \quad x_3 \quad x_3x_0 \quad x_3x_1 \\
& x_3x_1x_0 \quad x_3x_2 \quad x_3x_2x_0 \quad x_3x_2x_1 \quad x_3x_2x_1x_0]
\end{aligned} \tag{10}$$

Then

$$\begin{aligned}
f(x) =& \mathbf{B}_{RM} \cdot \mathbf{S}_f \\
=& 1 \oplus x_0 \oplus x_2 \oplus x_2x_1 \oplus x_2x_1x_0 \oplus x_3x_0 \oplus x_3x_2 \oplus x_3x_2x_0
\end{aligned} \tag{11}$$

This expression corresponds to the zero polarity, meaning that no variable is used complemented. A usual measure of the complexity of the expression is the number of its terms. In this case, this complexity is 8.

It is simple to see that $f(x)$ may also be written as:

$$
\begin{aligned}
f(x) =&1 \oplus x_0 \oplus x_2 \oplus x_2x_1 \oplus x_3x_0 \oplus x_3x_2 \oplus (x_1 \oplus x_3)x_2x_0 \\
f(x) =&1 \oplus x_0 \oplus x_2 \oplus x_3x_0 \oplus (x_1 \oplus x_3)(x_2x_0 \oplus x_2)
\end{aligned}
\tag{12}
$$

leading to the realization shown in Fig. 11, with a quantum cost of 20 and one ancillary line (besides the target line).

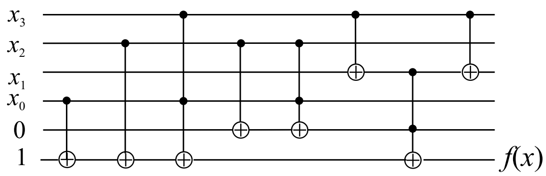


Fig. 11. Realization of the zero-polarity Reed Muller expression for $f(x)$.

Using the algorithm [13] it may be found that the best polarity is obtained when all four variables are complemented, leading to the following expression, with complexity 5.

$$f_{best-pol}(\boldsymbol{x}) = \overline{x}_2 \oplus \overline{x}_1\overline{x}_0 \oplus \overline{x}_2\overline{x}_1\overline{x}_0 \oplus \overline{x}_3 \oplus \overline{x}_3\overline{x}_2\overline{x}_0$$

It is relatively simple to deduce the following mixed polarity expression with complexity 4 and its equivalent hybrid expression:

$$
\begin{aligned}
f_{MPRM}(\boldsymbol{x}) =&x_2 \oplus x_3 \oplus x_2\overline{x}_1\overline{x}_0 \oplus \overline{x}_3\overline{x}_2\overline{x}_0 \\
f_{MPRM}(\boldsymbol{x}) =&x_2 \oplus x_3 \oplus (x_2\overline{x}_1 \oplus \overline{x}_3\overline{x}_2)\overline{x}_0 \\
f_{hybrid}(\boldsymbol{x}) =&x_2 \oplus x_3 \oplus (x_2\overline{x}_1 \oplus \overline{(x_3 \vee x_2)})\overline{x}_0 \\
f_{hybrid}(\boldsymbol{x}) =&x_2 \oplus x_3 \oplus (x_2\overline{x}_1 \oplus 1 \oplus (x_3 \vee x_2))\overline{x}_0
\end{aligned}
\tag{13}
$$

The hybrid expression has the realization shown in Fig. 12, with a quantum cost of 17 and one ancillary line besides the target line, taking advantage of some of the new gates introduced earlier.

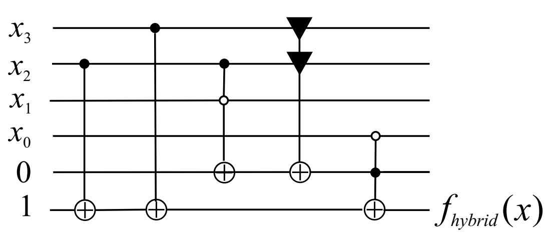


Fig. 12. Realization of the hybrid-MPRM expression.

Notice that if the output at the ancillary line is considered as garbage, then the last gate could be Peres-based and this would additionally reduce the quantum cost by 1.

It is interesting to point out that the *second best* polarity expression (with a complexity of 6 terms)- was obtained when complementing only x_0.

$$f_{2nd.best}(\boldsymbol{x}) = \overline{x}_0 \oplus x_2 \oplus x_3 \oplus x_3\overline{x}_0 \oplus x_1x_2\overline{x}_0 \oplus x_3x_2\overline{x}_0 \tag{14}$$

This second best expression leads however to an MPRM expression with complexity 5 and a realization, as shown in Fig. 13, with a quantum cost of 18 if CNOT and Toffoli gates are used. The quantum cost may be reduced to 17 if the last CCNOT gate is chosen to be a Peres gate.

$$\begin{aligned} f_{MPRM}(\boldsymbol{x}) =& \overline{x}_0 \oplus x_2 \oplus x_3x_0 \oplus (x_1 \oplus x_3)x_2\overline{x}_0 \\ =& \overline{x}_2 \oplus \overline{x}_3x_0 \oplus (x_1 \oplus x_3)x_2\overline{x}_0 \end{aligned} \tag{15}$$

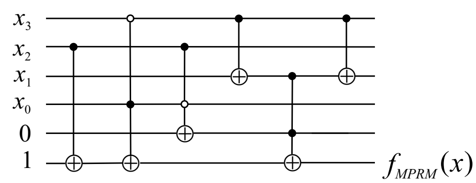


Fig. 13. Realization based on the second best polarity.

Finally, by complementing x_0 and x_3, one of the 3^{rd}. best expressions with complexity 7 may be obtained, which however leads to an MPRM expression *also of complexity 4* (as in the case of the best polarity) and identical to the best hybrid expression obtained from the second best polarity! (Compare Eqs. (15) and (17))

$$f_{3rd.best}(\boldsymbol{x}) = 1 \oplus x_2 \oplus \overline{x}_3 \oplus \overline{x}_3\overline{x}_0 \oplus x_2x_1\overline{x}_0 \oplus x_2\overline{x}_0 \oplus \overline{x}_3x_2\overline{x}_0 \tag{16}$$

$$\begin{aligned} f_{MPRM}(\boldsymbol{x}) =& \overline{x}_2 \oplus \overline{x}_3x_0 \oplus x_2\overline{x}_1\overline{x}_0 \oplus \overline{x}_3x_2\overline{x}_0 \\ =& \overline{x}_2 \oplus \overline{x}_3x_0 \oplus (x_1 \oplus x_3)x_2\overline{x}_0 \end{aligned} \tag{17}$$

Table 3. Summary of the main observations in example 2.

Polarity	Number of terms of the FPRM expression	Number of terms of the hybrid/MPRM expression	Quantum cost
0	8	—-	20
best	(5)	4	17/(16)
2^{nd}.*best*	(6)	4	18/(17)
3^{nd}.*best*	(7)	4	18/(17)

5 Conclusions

5.1 Remarks

In the classical FPRM transform, the last spectral coefficient corresponds to the odd-parity of the truth vector of the function being transformed, since the last row of the transform matrix is a row of 1s [17]. This means that the RM expression of a function with odd Hamming weight, will include the term corresponding to the conjunction of all variables, and this is a term with an expensive quantum realization. The same argument allows the same conclusion in the case of MPRM expressions. In the context of ESOP realizations the appearance of a term comprising the conjunction of all variables in the case of functions with an odd number of minterms has already been addressed in [8]. The inclusion of hybrid Reed Muller De Morgan expressions may however occasionally alleviate this problem.

For instance, the following function has a Hamming weight of 7:

$$f(x_3,x_2,x_1,x_0) = x_3x_2 \oplus x_1x_0 \oplus x_3x_2x_1x_0 = x_3x_2 \vee x_1x_0 \tag{18}$$

The GF(2) expression in Eq. (18) requires the realization of the 4-literals conjunction, which is quantum-expensive; meanwhile its Boolean equivalent only requires 2-literals conjunctions. The difference in quantum cost (if 2 ancillary lines besides the target line are allowed) is however only 2.

As illustrated in Example 2, an MPRM expression with lowest complexity, does not necessarily follow only from an FPRM expression with lowest complexity. MPRM expressions with minimal complexity do not necessarily have a realization with the same lowest quantum cost. The number of terms in the RM expression does not uniquely characterize the realization complexity (quantum cost). The distribution of complemented literals plays an important role and this is not specified by the number of terms of an MPRM expression.

5.2 Summary

In the design of reversible circuits, the use of negated control signals does not only simplify the representation of circuit schemes, but may also reduce the quantum cost. Furthermore, besides ANDing the control variables, the other basic connectives NAND, OR and NOR are applicable. These additional connectives also have a direct “Barenco-type” efficient quantum realization. In the case of some complemented control variables, the De Morgan laws should be observed to select the most economical realizations. Obtaining a circuit with white dots to operate with negated control signals is closely related to optimal Reed Muller expressions with mixed polarity. Formal methods to obtain minimal hybrid Reed Muller De Morgan expressions must however be developed.

The fact that the paper is based on Reed Muller expressions instead of ESOP expressions should not be understood as an implicit claim that results obtained via MPRM are better than results that might have been obtained via ESOP. It only reflects the higher familiarity of the author with RM than with ESOP. In the case of working with ESOP, also formal methods to obtain minimal hybrid ESOP De Morgan expressions would have to be developed.

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References

[1] R. Landauer, "Irreversibility and heat generation in the computing process," *IBM J. Res. Develop.*, vol. 5, pp. 183–191, 1961.

[2] C. H. Bennett, "Logical reversibility of computation," *IBM J. Res. Development*, vol. 17, pp. 525–532, 1973.

[3] A. Barenco, C. H. Bennett, R. Cleve, D. P. D. Vinchenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, "Elementary gates for quantum computation," *Phys. Rev.*, vol. A 52, pp. 3457–3467, 1995.

[4] A. Agrawal and N. K. Jha, "Synthesis of reversible logic," *Proc. DATE*, pp. 21 384–21 385, 2004.

[5] G. W. Dueck, D. Maslov, and D. M. Miller, "Transformation-based synthesis of networks of Toffoli/Fredkin gates," in *Proc. IEEE Canadian Conf. Electrical and Computer Engineering*, 2003, pp. 211–214.

[6] A. Mishchenko and M. Perkowski, "Logic of reversible wave cascades," *Proc. IWLS*, pp. 197–202, 2002.

[7] V. V. Shende, A. K. Prasad, I. L. Markov, and J. P. Hayes, "Synthesis of reversible logic circuits," *IEEE Trans. Computer Aided Design Integrated Systems*, vol. 22, no. 6, pp. 723–729, 2003.

[8] T. Sasao, *Switching Theory for Logic Synthesis*. Kluwer Academic Publishers, 1999, problem 13.12.

[9] D. M. Miller and G. W. Dueck, "Spectral techniques for reversible logic synthesis," in *Proc. 6th Int. Symp. Representations and Methodology of Future Computing Technologies*, 2003, pp. 56–62.

[10] C. Moraga and R. S. Stanković, "On a property of spectral transforms," in *Proc. Reed Muller Workshop*. Iizuka, Japan: World Printing Co., 2009, pp. 19–26.

[11] D. Green, "Reed Muller canonical forms with mixed polarity and their manipulations," *IEE Proc. E, Computer and Digital Techniques*, vol. 137, no. 1, pp. 103–113, 1990.

[12] A. Tran and J. Wang, "Decomposition method for minimization of Reed Muller polynomials in mixed polarity," *IEE Proc.*, vol. E, 140, no. 1, pp. 65–68, 1993.

[13] D. Janković, R. S. Stanković, and C. Moraga, "Optimization of polynomial expressions by using the extended dual polarity route," *IEEE Trans. Computers*, vol. 58, no. 12, pp. 1710–1725, 2009.

[14] D. Maslov, G. W. Dueck, D. M. Miller, and C. Negrevergne, "Quantum circuit simplification and level compaction," *IEEE Trans. CAD of Integrated Circuits and Systems*, vol. 27, no. 3, pp. 436–444, 2008.

[15] Z. Sasanian and D. M. Miller, "Transforming MCT circuits to NCVW circuits," in *Proc. 3rd Workshop on Reversible Computation*. Gent, Belgium: Press University of Bremen, July 2011, pp. 163–174.

[16] C. Moraga, "Low quantum cost realization of toffoli gates with multiple mixed control signals," European Centre for Soft Computing, 33600 Mieres, Spain, Tech. Rep. FSC 2011 – 08, 2011, Research report.

[17] J. T. Astola and R. S. Stanković, *Fundamentals of Switching Theory and Logic Design*. Heidelberg: Springer, 2006.