

Weak Attractors and Invariant Sets in Lorenz Model

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Abstract: A two-dimensional model is analyzed. It reflects the dynamics occurring in discrete Lorenz model. Invariant sets are analytically detected and the parameter space is investigated in order to classify completely regions of existence of stable 2-cycles, and regions associated with chaotic behaviors. This paper describes complex dynamics of invariant sets and weak attractors according to Tsybulin and Yudovich idea. These sets are displayed by numerical simulations.

Keywords: Noninvertible map; invariant set; weak attractor; bifurcation; basin of attraction.

1 Introduction

WE present and explain numerical results illustrating the mechanism of a type of bifurcation of a chaotic set that occurs in a typical dynamical system relative to discrete Lorenz mapping. Because the non-unique dynamics associated with extremely complex structures of the basin boundaries, the Lorenz model presents a real interest and a large richness of the bifurcations situations, and an interesting set of dynamical phenomena is uncovered, due in essence to the presence of invariant sets [1, 2]. This can have a profound effect on our understanding of the dynamical behavior.

We also provide numerical evidence of such a bifurcation for the appearance of invariant sets in this model.

Consider this dynamical system generated by a family of two-dimensional continuous noninvertible maps T_b defined by

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$$T_b : \begin{cases} x' = (1 + ab)x - bxy \\ y' = (1 - b)y + bx^2 \end{cases} \quad (1)$$

where a, b are real parameters, the functions $f(x, y) = (1 + ab)x - bxy$ and $g(x, y) = (1 - b)y + bx^2$ continuous and differentiable and T_b of type $Z_1 < Z_3$ in sense of Mira [for more details, see 2,3].

This map already studied by Lorenz. He predicted chaotic behavior when b is excessively large. His paper cited therein [3] is a milestone in the study of deterministic nonlinear dynamical systems. This fact has fundamental and known consequences. He illustrates the pertinence of the concept of computational chaos.

In two papers [1, 2], we have accomplished the task to illustrate the fractal basins and to consider the inverses with vanishing denominators and we studied the properties of such mappings in our examination of developed chaos. All these behaviors displayed homoclinic structure associated with the basin bifurcation bounded-nonbounded. We know that chaos exists in any system with a homoclinic tangency, it follows that it can be found in the space of parameters of any chaotic dynamical model in absence of uniform hyperbolicity, in particular in such popular examples as Hénon map, Chua circuit, Lorenz model, etc. Therefore, the problem of understanding the nature of the orbit structure for systems from invariant regions is quite challenging. In these papers our investigation of fractality as a measure of chaos consisted to analyze the fractalization and parameter dependence of basins by using the technique of critical curves.

The description of the dynamics of such systems requires an infinite set of invariants which means that any attempt to give a complete description of the dynamics and bifurcations will fail. Therefore, we have to restrict the analysis to some particular details or to some most general features only.

Tsybullin and Yudovich [4] considered quadratic mappings as a finite-difference approximation of an ordinary differential system and determined interesting invariant manifold and sets, and pointed out the invariant measure on the invariant disc. The notion of invariant curves, introduced in [4], constitutes an analytical instrument particularly suitable to study the dynamical behavior and bifurcations in two-dimensional maps with nonunique inverse. Its task is to provide reasonable and rigorous explanations for our investigation.

The main topic of this paper is motivated by our desire to apply and use ideas and methods of Tsybullin & Yudovich to illustrate complex patterns of Lorenz model proposed in [3]. It was mainly focused on one research area, to detect weak attractors and invariant sets, and to identify and verify some properties of attractors on such maps. For the needs of bifurcation theory, it is very important to guarantee

the persistence of attractors under perturbations by higher order terms in x and y .

Countless papers have been published in the past on the existence of hypersurface of a mapping of some manifold, in which it is proved that the existence is rare and very degenerate situation. Arnold in [5] considered that the absence of such invariant hypersurfaces as one of the definitions of strong nonintegrability and conjured this phenomena as the case of some open problem in the space of maps.

This paper is organized as follows. The section 2 recalls some peculiar properties of the Lorenz map, their dependence on the parameters is considered, and stability of a peculiar fixed point 'the origin' is analyzed. The qualitative behavior and bifurcations of this map are examined. In section 3, We investigate cycles of order 2, and we examine the special case of $b = 2$. We discuss some cases where bifurcation can lead to creation of holes in basins of attraction, and cause qualitative changes in the structure of the domain as some parameters are varied. In section 4, we consider conditions for the existence of invariant sets. This new kind of invariant set is of mixed type in the sense that it is made up of curves invariant by T and T^2 which constitute weak attractors. And finally, we give the conclusion.

2 Fixed points and Critical curves

The fixed point and basic bifurcations of T of the eq. (1) were analyzed, they are solutions obtained by a trivial manipulation of (1) with $x' = x$ and $y' = y$: Besides the solution $(0;0)$, we can observe that further fixed points exist if $a > 0$. There have been many important and interesting results about this system, such as the global stability, attractors, basins and so on.

In this section we focus our attention on bifurcations playing an important role in the dynamics, those happening for $a > 0$ and $b > 0$. We can easily state the following proposition.

Proposition 1 *If $a < 0$ then $O(0;0)$ is the unique fixed point of the map T_b defined by (1). If $a > 0$ then two further fixed points, P and P_0 , exist, symmetric with respect to the y -axis, with $x = \pm\sqrt{a}$; $y = a$.*

We consider the qualitative behaviors of the map (1). As usual, the local dynamics of map (1) in a neighborhood of a fixed point is dependant on the Jacobian matrix. The Jacobian matrix of map (1) is given by

$$J = \begin{bmatrix} 1 + ab - by & -bx \\ 2bx & 1 - b \end{bmatrix}$$

We consider now the conditions of local stability of the fixed point $O(0;0)$, in terms of the parameters of the eq. (1).

With

$$J_{(0,0)} = \begin{bmatrix} 1 + ab & 0 \\ 0 & 1 - b \end{bmatrix}$$

is Jacobian matrix of T_b in $O(0;0)$ which has two eigenvalues $1 + ab$ and $1 - b$. We can conclude for $b > 0$ with considering $a > 0$ and the case $a = 0$ that there are only two different topological types of $O(0;0)$ for all permissible parameter values. The fixed point $O(0;0)$ is a saddle if $b \in [0;2]$. When $a > 0$, $O(0;0)$ is a source if $b \in [2;\infty)$ and when $a > 0$, $O(0;0)$ is non hyperbolic if $a = 0$. We can see that when $b = 2$, for $a > 0$, one of the eigenvalues of $O(0;0)$ is -1 and the other is not one with module. Thus the flip bifurcation occurs with a birth of a pair of saddle-cycles of order 2. Local stability of fixed point $O(0;0)$ in Table 1 is given.

Table 1. Local stability of the fixed point $O(0;0)$.

$b(a > 0)$	$[0;2]$	$[2;\infty)$
$\lambda_1 = 1 - b$	$-1 < \lambda_1 < 1$	$\lambda_1 < -1$
$\lambda_2 = 1 + ab$	$\lambda_2 > 1$	$\lambda_2 > 1$
$O(0;0)$	saddle	unstable node
$b (a = 0)$	$[0;2]$	$[2;\infty)$
$\lambda_2 = 1 + ab$	$\lambda_2 = 1$	$\lambda_2 = 1$
$O(0;0)$	saddle \longrightarrow Unstable node	unstable node \longrightarrow saddle

We can obtain that there are only two different topological types of $O(0;0)$ for all permissible parameter values.

Proposition 2 *The point $O(0;0)$ is a saddle if $b < 2$, $a > 0$; $O(0;0)$ is a unstable node if $b > 2$, $a > 0$.*

Proposition 3 *If $a = 0$, the map (1) undergoes a pitchfork bifurcation at $O(0;0)$.*

Proof: By simple computation, we can prove this proposition.

Proposition 4 *For $a > 0$ and $b = 2$, the map (1) undergoes a basin bifurcation bounded-nonbounded.*

This implies that bifurcations of such systems are extremely involved and rich. Thus, the question arises: Can we estimate in exact terms how rich these bifurcations are?

3 Continuous family of 2-cycles order

For $b = 2$, examining the properties of cycles, we remark that cycles of order 2 are detectable on the whole y -axis excepting the fixed point at the origin $y = 0$. These cycles $C_2: \{(0, y), (0, -y)\}, y \neq 0$ are transversally stable and according to known results in Tsybullin [4] on stability of continuous families, imply that if their transversal multiplier $|\rho_{\perp}| > 1$ then C_2 are unstable. Longitudinal multiplier ρ_{\top} is equal to 1.

Since $\rho_{\perp} = (1 + 2a)^2 - 4y^2$, stable cycles $C_2: \{(0, y), (0, -y)\}$ belong to this pair of intervals $\{y : 2a\sqrt{1 + 1/a} < y < \sqrt{(1 + 2a)^2 + 1}\}$

For $b = 2.002$ ($a = 0.1$), Fig. 1 shows that we are close to the bifurcation value at $b = 2$ that leads to basin transformation from a bounded basin to an unbounded basin for $b < 2$. The saddle fixed point at $O(0, 0)$ has become an unstable node, with the birth of a pair of 2-cycles of the saddle type, symmetric with respect to the y -axis.

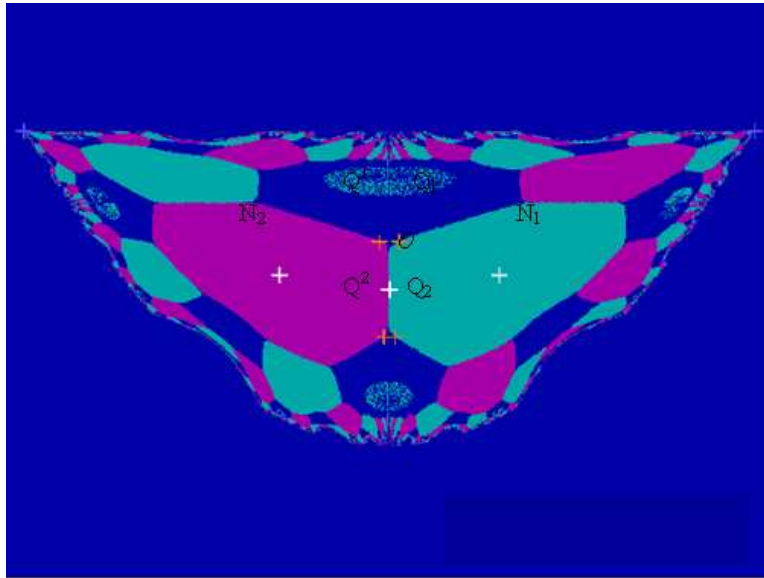


Fig. 1. Basin bifurcation and the origin is a node.

If we increase the parameter $a = 0.437$ for $b = 2$, we observe interesting situation and we obtain characteristic behaviors of riddled basins. These basins appear because symmetries of dynamical systems force the presence of invariant submanifolds; the attractors within invariant manifolds may be only weakly attracting transverse to the invariant manifold and this leads to a basin structure that is, roughly

speaking, full of holes. By examining this peculiar value of b , greater understanding can be gained as to how such basins arise.

It is known that basins generated by two-dimensional noninvertible maps may be either simply connected, or multiply connected, or non connected, depending on the situation of their boundary with respect to the critical set LC . In particular, in Mira [3], the concept of minimal invariant absorbing area is defined in order to give a global characterization of the different dynamical scenarios related to riddling bifurcation.

The minimal invariant absorbing area is the smallest absorbing area that includes the Milnor attractor [6] on which the chaotic dynamics occurs. Its delimitation is important in order to characterize the global properties which influence the qualitative effects of riddling bifurcation. The period-1 saddle embedded in the absorbing area which contains the attractor becomes transversely unstable via a supercritical period doubling bifurcation, leading to the birth of two new period-2 saddles situated on the tops or extremities belonging to y – axis. In addition, Fig. 2 displays basin in yellow color and weak attractor (in red color) made of bands. Whenever the origin of saddle type changes in an unstable node for this choice of b producing the desired result: holes in the basin and offers the information of riddled basin and 2-cycles are generically born which with the symmetric appearance will give two 2-cycles of saddle type for $b > 2$.

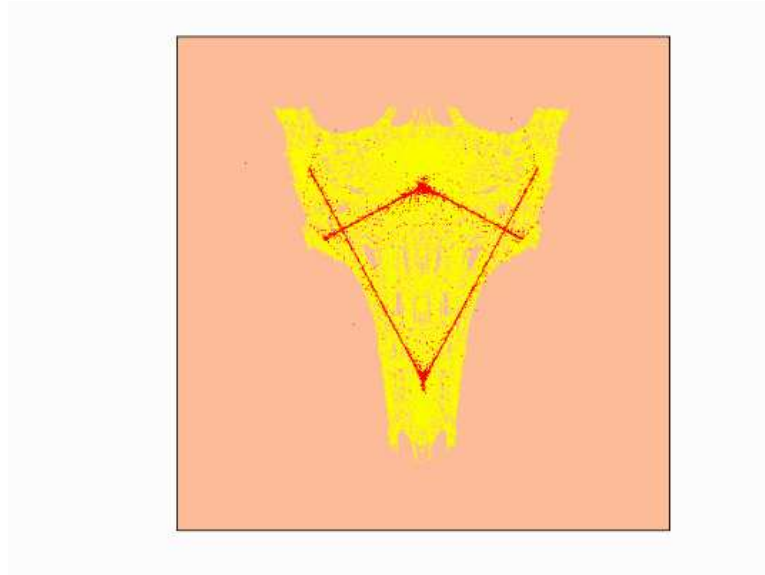


Fig. 2. Weak Attractor and Riddled basin.

4 T-Invariant sets and T^2 -invariant sets

In this part, Invariant sets generated by two-dimensional endomorphisms are studied. These invariant sets are obtained by iterating invariant lines in the immediate bounded basin. This new kind of invariant set is of mixed type in the sense that it is made up of curves invariant by T and T^2 which constitute weak attractors.

Attractors constitute an interesting object of study by themselves. The strong dependence on the parameters generates a rich variety of complex patterns on the plane and gives rise to different types of basin fractalization as a consequence, for instance, of basins bifurcations with holes [1, 2]. Taking into account the complexity of the matter and its nature, the study of these phenomena can be carried out only via the association of numerical investigations guided by fundamental considerations that can be found in [4].

The case $b = 2$ is very interesting, because we can put in evidence the existence of weak attractors. The special character of this kind of sets has been already observed in coupled maps. To illustrate the idea, consider first invariant algebraic curves of the first order (invariant lines), of second order (invariant crosses) in sense of Tsybullin [4].

Definition 1 (from [4]): Let $T : R^2 \rightarrow R^2$, be a map of R^2 into itself. The invariant cross is a union $L_1 \cup L_2$ of two subsets of R^2 such that $T(L_1) \subset L_2$ and $T(L_2) \subset L_1$. We denote it $L_1 L_2$, and L_1 and L_2 invariant with respect to T^2 so that $T^2(L_1) \subset L_1$ and $T^2(L_2) \subset L_2$.

The mapping (1) has several invariant crosses. We start by determining the straight crosses, so that L_1 and L_2 are the lines:

$$\text{Put } L_1 : y = \alpha x + \beta \quad ; \quad L_2 : y = \gamma x + \delta$$

In virtue of the property $T(L_1) \subset L_2$ and $T(L_2) \subset L_1$, we get:

$$\begin{aligned} (1-b)(\alpha x + \beta) + bx^2 &= \gamma(1+ab)x - \gamma bx(\alpha x + \beta) + \delta \\ (1-b)(\gamma x + \delta) + bx^2 &= \alpha(1+ab)x - \alpha bx(\gamma x + \delta) + \beta \end{aligned}$$

From these two inequalities, we have: $\alpha = -1/\gamma$; $\delta = -\beta$ and $\alpha^2 = 1 + 2a - 2\beta$. Then for $b = 2$, we have two crosses $L_1^+ L_2^+$ and $L_1^- L_2^-$ such that:

$$\begin{aligned} L_1^\pm : y &= \alpha_\pm x + \beta \quad \text{and} \quad \alpha^2 = 1 + 2a - 2\beta; \\ L_2^\pm : y &= \gamma_\pm x + \delta \quad \text{and} \quad \gamma^2 = 1 + 2a - 2\delta \end{aligned}$$

The restriction of the mapping T^2 to each line so that $T^2(L_1) \subset L_1$ gives: $(1 + 2a)^2 - 4l^2 = 1$.

In the other hand, if we search T^2 -invariant lines by considering equation $y = c$ and T^2 -mapping which acts to the rule: $T^2 : (x, y) \longrightarrow (x', y')$ with

$$x' = (1 + ab)[(1 + ab)x - bxy] - b[(1 + ab)x - bxy][(1 - b)y + bx^2]$$

and

$$y' = (1 - b)[(1 - b)y + bx^2] + b[(1 + ab)x - bxy]^2$$

we obtain

$$c = (1 - b)((1 - b)c + bx^2) + bx^2((1 + ab) - bc)^2, \quad \forall x \in \mathbb{R}$$

Then: $c = (1 - b)^2 c$; $b(1 - b) + b((1 + ab) - bc)^2 = 0$

If $b = 2$, T^2 -invariant lines are $S_{\pm} : y = (1 + 2a \pm 1)/2$, whose preimages under iteration T are T^2 invariant parabolas, and these lines S_{\pm} are transversally unstable such that S_+ corresponds to the line $y = 1 + a$, and $S_- : y = a$. Fig. 3 shows invariant curved crosses $L_{1,2}^{\pm}$, parabolas and S_{\pm} for $a = 0.42$ and $b = 2$.

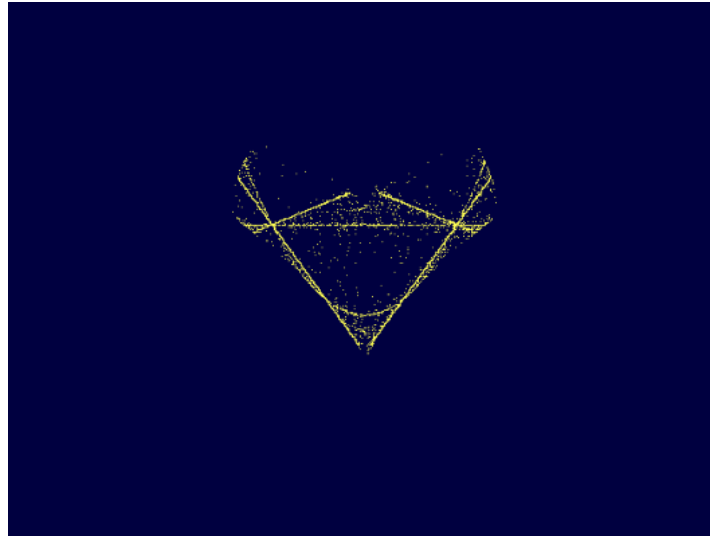


Fig. 3. Invariant sets parabolas and lines.

These segments are also evoked in detail in [2], for instance, the existence of the curve-line $y = 1 + a$ mapped into the origin makes the iteration properties of T (for $b = 1$) very different with respect to the behavior of the maps with a unique inverse. Any arc crossing this line is mapped into an arc with a loop in the origin like seen in [5]. Since the origin is a saddle, This requires that this curve belongs to the stable manifold of this point. Its role is important to understand the homoclinic

bifurcation of $O(0,0)$, giving rise to a unique chaotic attractor which intersects the line $y = 1 + a$ in infinitely many arcs with self-similar structure and loops issuing from the origin (see Fig. 4 for $b = 0.4, a = 2$).

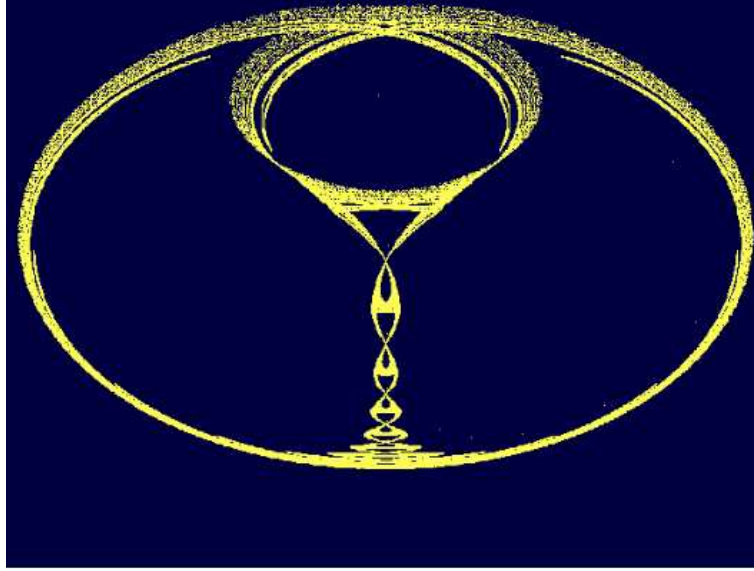


Fig. 4. Homocline Bifurcation and Loops in S_{\pm} .

Consider the Jacobian matrix of ToT

$$\begin{bmatrix} [(1+ab) - by] & -b(1+ab)x \\ \times [(1+ab) - by(1-b) - 3b^2x^2] & -b(1+ab)(1-b) + 2b(1-b)yx + b^3x^3 \\ 2(1-b)bx + 2bx[(1+ab) - by]^2 & (1-b)^2 - 2b^2x^2[(1+ab) - by] \end{bmatrix}$$

We remark that $\rho_{\top} = (1-b)^2 - 2b^2x^2[(1+ab) - by]$, and if we consider the 2-cycles $(0, \pm y)$ for $b = 2$ we obtain $\rho_{\top} = 1$. Further, $\rho_{\perp} = [(1+ab) - by][(1+ab) - by(1-b) - 3b^2x^2]$, in these 2-cycles $(0, \pm y)$ for $b = 2$, is equal to $\rho_{\perp} = (1+2a)^2 - 4y^2$.

These results may be interesting, since the delimitation of the invariant sets and basins permits to understand and clarify the concept of weak attractors. We provide numerical evidence of such a bifurcation for chaotic sets of the Lorenz model.

5 Conclusion

In this paper we investigate invariance property of the discrete Lorenz model. In contrast to existing studies of such models, which primarily focus on the dynamic

complexities in the models or the identification of parameter regions, our work gives another perspective of the global dynamical behavior by a study of invariant sets and their bifurcations. We also give an explanation for the structure of weak attractors by use of the invariant set. These chaotic sets are related to the model itself, and the main qualitative changes of the global structure can be analyzed by the contact between the domain boundaries and the singular set for the degenerated map. This allows us to learn more about the dynamical behavior of the systems under study than by just focusing on the local dynamics, since we can obtain sets specific to coupled maps.

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