Dynamics of Three Dimensional Maps

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Abstract: Smooth 3D maps have been a focus of study in a wide range of research fields. Their Properties are investigated qualitatively and numerically. These maps show qualitatively interesting types of bifurcations than those exhibited by generic smooth planar maps. We present a theoretical framework for analyzing three-dimensional smooth coupling maps by finding the stability criteria for periodic orbits and characterizing the system behaviors with the tools of nonlinear dynamics relative to bifurcation in the parameter plane, invariant manifolds, critical manifolds, chaotic attractors. We also show by numerical simulation bifurcations that can occur in such maps. By an analytical and numerical exploration we give some properties and characteristics, since this class of three-dimensional dynamics is associated with the properties of one-dimensional maps. There is an interesting passage from the one-dimensional endomorphisms to the three-dimensional endomorphisms.

Keywords: Three-dimensional maps, bifurcations, invariant closed curve.

1 Introduction

Three parameters bifurcation problem is not frequently used for analyzing nonlinear dynamical systems. Somes pecular dynamical properties have been evidenced and observed in iterated maps of IR^2. There has been an explosion of research activity concerned with chaotic behavior and then many books on dynamical systems to reflet the recent interest, but relatively few of the books to offer a large account of the area of three-dimensional maps. The essence of scientific efforts is shifted to further elaboration of conceptual framework of bifurcation analysis, to standardization of the new important domains of applications for the description the qualitative properties of orbits. The basic element of this analysis is the geometrical and numerical modification and application of the classical formalism, which is

giving the description of the behavior of the iteration processes near the boundaries of the stability domains of equilibria. Our present work attempts towards finding suitable stability criteria of periodic orbits in three-dimensional smooth systems with respect to certain parameters in the map, which is derived on the parts on the parameter-scannings. Different bifurcation scenarios and existence of chaotic attractors are also shown by computer simulation.

2   Presentation

The starting point for us, was two-dimensional smooth maps of the form:

$$T_0 : \mathbb{R}^2 \to \mathbb{R}^2$$

$$T_0 : \begin{cases} X_{n+1} = 4.a_1.y_n.(1 - y_n) + (1 - b).x_n \\ Y_{n+1} = 4.a_2.x_n.(1 - x_n) + (1 - b).y_n \end{cases}$$

Where a_1, a_2, b are real parameters. Classic bifurcations were put in evidence for these maps related to critical curves , to chaotic attractors and basins. These bifurcations are the following.

a. Connected Basin $\longleftrightarrow$ Multiply connected basin

b. Non connected Basin $\longleftrightarrow$ connected basin

c. Contact bifurcation and disparition of an attractor

d. Fractalization of the basin boundary

e. Invariant Closed Curve (ICC) $\longleftrightarrow$ Attractor Weakly Chaotic (AWC) : transformation of an invariant closed curve in weakly chaotic attractor.

f. Contact bifurcations of chaotic areas.

Developing and exploring non linear maps in 3-dimension extended from T_0 is a natural research topic. We consider the extended form T_1 as follows:

$$T_1 : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T_1 : \begin{cases} x_{n+1} = 4.a_1.y_n.(1 - y_n) + (1 - b).x_n \\ y_{n+1} = 4.a_2.z_n.(1 - z_n) + (1 - b).y_n \\ z_{n+1} = 4.a_3.x_n.(1 - x_n) + (1 - b).z_n \end{cases} \qquad (1)$$

where a_1, a_2, a_3, b are real parameters and x, y, z represent the space. We must notice that the different choices of parameters give a wide variety of dynamical behaviors. The dynamics involves various transitions by bifurcations.

Our new 3D map T_1illustrates important routes to chaos related to Neimark-Hoph bifurcation, to doubling bifurcation. This has provided the principal motivation for the present work . Indeed, analogous phenomena concerning k-cycles produce invariant closed curves.

This paper intends to give such a study, and to consider different maps. It is structured as follows. First; Section 2, gives some general properties and bifurcations are proved. Since the three-dimensional dynamics of T_2 are associated with the properties of a one-dimensional map, there is an interesting passage from the one-dimensional endomorphism to the three-dimensional endomorphism, and then some properties are automatically deduced. Next; in Section3, we introduce the notation used in [1, 2] to analyze these maps for a new kind of bifurcation, which is also a new route to chaos, and we note basic definitions and facts about this kind of maps. Conclusions are given in section 4.

3   Study of the case b=1

In this section, in our study of three-dimensional maps we investigate our singularities using some techniques and numerical simulations. Due to the theoretical and practical difficulties involved in the study, computers will presumably play a role in such efforts. We study now the system T_2 with $b = 1$, $a_1, a_2, a_3 \in \mathbb{R}$. We start by the simple case and we develop this work

$$T_2 : \begin{cases} x_{n+1} = 4.a_1.y_n.(1-y_n) \\ y_{n+1} = 4.a_2.z_n.(1-z_n) \\ z_{n+1} = 4.a_3.x_n.(1-x_n) \end{cases} \tag{2}$$

where a_1, a_2, a_3 are real parameters.

First, we present the diagram of bifurcations in the parameter plane (a_1, a_3), and we describe the dynamic behavior of T_2. With this scanning, a meaningful characterization occurs and consists in the identification of its singularities, and its dynamical behavior as the parameters vary. The numerical procedure of the description of such phenomena includes the bifurcation diagrams in which the bifurcation parameter is the equilibrium itself. On the other hand, we obtain informations on stability region for the fixed point (blue domain), and the existence region for attracting cycles of order k exists ($k \leq 14$). The black regions ($k = 15$) corresponds to the existence of bounded iterated sequences. Parameters lying in different regions give rise to different kind of bifurcations depending on the stability of the existing k-cycle.
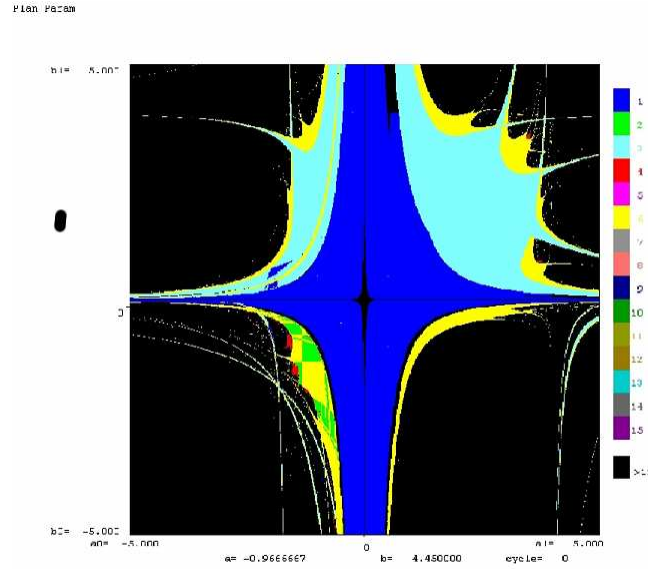
Fig. 1. Bifurcation diagram for T_2 in the parameter plane (a_1, a_3)

3.1 Simple generalization

The map T_2 can be written in the following form:

$$T_2^*(x,y,z) : \begin{cases} x_{n+1} = f(y_n) \\ y_{n+1} = g(z_n) \\ z_{n+1} = h(x_n) \end{cases} \tag{3}$$

where $f : Y \to X$, $g : Z \to Y$ and $h : X \to Z$ are continuous maps. With the initial condition $(x_0, y_0, z_0) \in X \times Y \times Z$ and a trajectory $\{x_t, y_t, z_t\}$, $t \geq 0$, where T_2^{*t} is the tth iterate of the map T_2^*. We shall construct the analytical representation of the general procedure of linear bifurcation analysis which describes the changes in the qualitative properties of the orbits on non-linear discrete dynamics under the changes of the parameters of these dynamics. A more generalized study of the T_2^* system has been done on basis of the classical oligopoly model [3] and of the two -dimensional case $(x_{n+1} = f(y_n), y_{n+1} = g(x_n))$, see in [4] , [5] and [6]. Consider global dynamics. Let us then define three functions F, G, H such that we can assume that :

$$F = f \circ g \circ h, \qquad G = g \circ h \circ f \quad and \quad H = h \circ f \circ g \tag{4}$$

where the sets X, Y and Z are assumed such that the maps F, G and H are well defined. Let's announce some properties of such maps. Very briefly we have the following:

Property 1: For any initial condition (x_0, y_0, z_0), these assumptions hold

$$T_2^{*3k}(x_0, y_0, z_0) \rightarrow (x_{3k} = F^k(x_0), y_{3k} = G^k(y_0), z_{3k} = G^k(z_0) \tag{5}$$

$$\begin{aligned} T_2^{*3k+1}(x_0, y_0, z_0) \rightarrow (x_{3k+1} &= f \circ G^k(y_0), \\ y_{3k+1} &= g \circ H^k(z_0), \\ z_{3k+1} &= h \circ F^k(x_0). \end{aligned} \tag{6}$$

$$\begin{aligned} T_2^{*3k+2}(x_0, y_0, z_0) \rightarrow (x_{3k+2} &= f \circ g \circ H^k(z_0), \\ y_{3k+2} &= g \circ h \circ F^k(x_0), \\ z_{3k+1} &= h \circ f \circ G^k(y_0). \end{aligned} \tag{7}$$

where $k = 1, 2, ..., F^k, G^k, H^k$ are the k iterate of F, G, H.

Property 2 : For each $k \geq 1$ the maps F, G and H satisfy:

$$f \circ G^k = f \circ g \circ h \circ f \circ ... \circ g \circ h \circ f = F^k \circ f$$
$$g \circ H^k = g \circ h \circ f \circ g \circ ... \circ h \circ f \circ g = G^k \circ g$$
$$h \circ F^k = h \circ f \circ g \circ h \circ ... \circ f \circ g \circ h = H^k \circ h$$

Property 3 : For each $k \geq 1$ the maps F, G and H satisfy:

$$f \circ g \circ H^k = f \circ g \circ h \circ f \circ g \circ ... \circ h \circ f \circ g = F^k \circ f \circ g$$
$$g \circ h \circ F^k = g \circ h \circ f \circ g \circ h \circ ... \circ f \circ g \circ h \circ = G^k \circ g \circ h$$
$$h \circ f \circ G^k = h \circ f \circ g \circ h \circ f \circ ... \circ g \circ h \circ f = H^k \circ h \circ f$$

Property 4 :

If $\{x_1, x_2, ..., x_k\}$ is a $k-$cycle of F then $\{z_1, z_2, ..., z_k\} = \{h(x_1), h(x_2), ..., h(x_k)\}$ is a $k-$ cycle of H.

If $\{y_1, y_2, ..., y_k\}$ is a $k-$cycle of G then $\{x_1, x_2, ..., x_k\} = \{f(y_1), f(y_2), ..., f(y_k)\}$ is a $k-$cycle of F.

If $\{z_1, z_2, ..., z_k\}$ is a $k-$cycle of H then $\{y_1, y_2, ..., y_k\} = \{g(z_1), g(z_2), ..., g(z_k)\}$ is a $k-$cycle of G

3.2 Study of fixed points and cycles

Proposition 3.1. *A fixed point (A_0, B_0, C_0) of T_2^* is constructed from a fixed point A_0 of F, a fixed point B_0 of G and a fixed point C_0 of H.*

Proof. (A_0, B_0, C_0) is a fixed point of $T_2^* \Rightarrow A_0$ a fixed point of F, B_0 and C_0 are fixed points of G and H respectively.

$$T_3^*(A_0, B_0, C_0) = (A_0, B_0, C_0) \Rightarrow T_2^* \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f(B_0) \\ g(C_0) \\ h(A_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} f(g(C_0)) \\ g(h(A_0)) \\ h(f(B_0)) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f(g(h(A_0))) \\ g(h(f(B_0))) \\ h(f(g(C_0))) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} F(A_0) \\ G(B_0) \\ H(C_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

$$T_2^*(A_0, B_0, C_0) = (A_0, B_0, C_0) \Rightarrow T_2^* \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f(B_0) \\ g(C_0) \\ h(A_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} f(g(C_0)) \\ g(h(A_0)) \\ h(f(B_0)) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} f(g(h(A_0))) \\ g(h(f(B_0))) \\ h(f(g(C_0))) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} F(A_0) \\ G(B_0) \\ H(C_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

Let A_0 be a fixed point of F, B_0 a fixed point of G and C_0 a fixed point of T_2^*, then (A_0, B_0, C_0) is a fixed point of T_2^* :

$$\begin{pmatrix} F(A_0) \\ G(B_0) \\ H(C_0) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} f(g(h(A_0))) \\ g(h(f(B_0))) \\ h(f(g(C_0))) \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow \begin{pmatrix} f[g(h(A_0)] \\ g[h(f(B_0))] \\ h[f(g(C_0))] \end{pmatrix} = \begin{pmatrix} A_0 \\ B_0 \\ C_0 \end{pmatrix}$$

we put

$$\begin{pmatrix} g(h(A_0) = B_0 \\ h(f(B_0)) = C_0 \\ f(g(C_0)) = A_0 \end{pmatrix} \Rightarrow \begin{pmatrix} cf(B_0) \\ g(C_0) \\ h(A_0) \end{pmatrix} = \begin{pmatrix} cA_0 \\ B_0 \\ C_0 \end{pmatrix} \Rightarrow T_2^* \begin{pmatrix} cA_0 \\ B_0 \\ C_0 \end{pmatrix} = \begin{pmatrix} cA_0 \\ B_0 \\ C_0 \end{pmatrix} \qquad (8)$$

$\square$

The stability of these points is naturally of fundamental importance. By linearizing of the fixed point $P(A,B,C)$ of T_2^*. The Jacobian matrix T_2^* at P is

$$J(A,B,C) = \begin{bmatrix} 0 & \dfrac{\delta f(B)}{\delta y} & 0 \\ 0 & 0 & \dfrac{\delta g(C)}{\delta z} \\ \dfrac{\delta h(A)}{\delta x} & 0 & 0 \end{bmatrix}$$

We then get, by expanding the determinant

$$J(A,B,C) = \lambda^3 + \frac{\delta f(B)}{\delta y} \cdot \frac{\delta g(C)}{\delta z} \cdot \frac{\delta h(A)}{\delta x}$$

Then we have three cases to consider: $n = 3k$, $n = 3k+1$, and $n = 3k+2$. The points of the cycles of order $n = 3k$ are described with the following expression:

$$\begin{cases} F^k(A) = A \\ G^k(B) = B \\ H^k(C) = C \end{cases}$$

where (A,B,C) is a cycle of order k. The jacobian matrix is given by

$$J^{3k}(A,B,C) = \begin{bmatrix} \dfrac{\delta F^k(A)}{\delta x} & 0 & 0 \\ 0 & \dfrac{\delta G^k(B)}{\delta y} & 0 \\ 0 & 0 & \dfrac{\delta H^k(C))}{\delta z} \end{bmatrix}$$

And the eigenvalues are all real:

$$\lambda_1 = \frac{\delta F^k(A)}{\delta x} , \lambda_2 = \frac{\delta G^k(B)}{\delta y} \text{ et } \lambda_3 = \frac{\delta H^k(C))}{\delta z}$$

The cycles of order $n = 3k$ ($k \geq 1$) can be nodes or saddles.

For the second case: The points of the cycles of order $n = 3k+1$ are described by:

$$\begin{cases} f \circ G^k(B) = A \\ g \circ H^k(C) = B \\ h \circ F^k(A) = C \end{cases}$$

The jacobian matrix related at this case is expressed by

$$J^{3k+1}(A,B,C) = \begin{bmatrix} 0 & \frac{\delta G^k(B)}{\delta y} \cdot \frac{\delta f[G^k(B)]}{\delta y} & 0 \\ 0 & 0 & \frac{\delta H^k(C))}{\delta z} \cdot \frac{\delta g[H^k(C)]}{\delta z} \\ \frac{\delta F^k(A)}{\delta x} \cdot \frac{\delta h[F^k(A)]}{\delta x} & 0 & 0 \end{bmatrix}$$

The characteristic equation for this kind of cycles is given by :

$$\lambda^3 + \frac{\delta f[G^k(B)]}{\delta y} \cdot \frac{\delta g[H^k(C)]}{\delta z} \cdot \frac{\delta h[F^k(A)]}{\delta x}$$
$$\cdot \frac{\delta G^k(B)}{\delta y} \cdot \frac{\delta H^k(C))}{\delta z} \cdot \frac{\delta F^k(A)}{\delta x} = 0$$

Therefore we have three solutions: $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 \in C$. The cycles of order $k = 3k+1$ are either nodes-focus, or saddles-focus. The last case: cycles of order $n = 3k+2$ verify this type of relation:

$$\begin{cases} f \circ g \circ H^k(C) = A \\ g \circ h \circ F^k(A) = B \\ h \circ f \circ G^k(B) = C \end{cases}$$

The jacobian matrix is

$J^{3k+1}(A,B,C) =$

$$\begin{bmatrix} 0 & 0 & \frac{\delta f[g[H^K(C)]]}{\delta z} \cdot \frac{\delta g[H^k(C)]}{\delta z} \cdot \frac{\delta H^k(C)}{\delta z} \\ \frac{\delta g[h[F^K(A)]]}{\delta x} \cdot \frac{\delta h[F^k(A)]}{\delta x} \cdot \frac{\delta F^k(A)}{\delta x} & 0 & 0 \\ 0 & \frac{\delta h[f[G^K(B)]]}{\delta y} \cdot \frac{\delta f[[G^k(B)]]}{\delta y} \cdot \frac{\delta G^k(B)}{\delta y} & 0 \end{bmatrix}$$

The equation of the eigenvalues is then:

$$\lambda^3 + \frac{\delta f[g[H^K(C)]]}{\delta z} \cdot \frac{\delta g[h[F^K(A)]]}{\delta x} \cdot \frac{\delta h[f[G^K(B)]]}{\delta y} \cdot \frac{\delta g[H^k(C)]}{\delta z} \cdot \frac{\delta h[F^k(A)]}{\delta x}$$
$$\cdot \frac{\delta f[[G^k(B)]]}{\delta y} \cdot \frac{\delta H^k(C)}{\delta z} \cdot \frac{\delta F^k(A)}{\delta x} \cdot \frac{\delta G^k(B)}{\delta y}$$

Here also we have three eigenvalues $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 \in C$. The cycles of order $3k+2$ are either nodes-focus, or saddles-focus.

3.3 Critical planes

The map T_2 is not-invertible. An important tool used to study non-invertible maps is that of critical manifold, which has been introduced by Mira [7] and [8] .

A non-invertible map is characterized by the fact that a point in the state space can possess different number of rank-one preimages, depending where it is located in the state space. In the three-dimensional case, a critical plane *PC* is the geometrical locus, in the state $J(X)$ space of points X having two coincident primages, $T^{-1}(X)$, located on a plane PC_{-1} . It is recalled that the set of points $T^{-n}(X)$ constitutes the rank-n preimages of a given point X.

For the map T_2 , The plane $PC = T_2(PC_{-1})$. The plane PC_{-1} is verifying $|J(X)| = 0$, where $J(X)$ is the jacobian matrix of T_2 at the point X which satisfies the equation

$$J(X) = \begin{bmatrix} 0 & 4a_1(1-2y) & 0 \\ 0 & 0 & 4a_2(1-2z) \\ 4a_3(1-2x) & 0 & 0 \end{bmatrix}$$

$$J(X) = 64.a_1.a_2.a_3.(1-2x)(1-2y)(1-2z)$$

We can remark that PC_{-1} is independent of the parameters. PC_{-1} is constituted of three planes: $PC_{-1}^{(a)}$, $PC_{-1}^{(b)}$, $PC_{-1}^{(c)}$, where $PC_{-1}^{(a)} = \{(x,y,z)\backslash x = \frac{1}{2}\}$, $PC_{-1}^{(b)} = \{(x,y,z)\backslash y = \frac{1}{2}\}$, $PC_{-1}^{(c)} = \{(x,y,z)\backslash z = \frac{1}{2}\}$

It follows that the critical planes of rank-1 are:

- $PC^{(a)} = T_2(PC_{-1}^{(c)}$) is the plane defined by $z = a_3$ with $y \le a_2$.

- $PC^{(b)} = T_2(PC_{-1}^{(b)}$) is the plane defined by $x = a_1$ with $z \le a_3$.

- $PC^{(c)} = T_2(PC_{-1}^{(c)}$) is the plane defined by $y = a_3$ with $x \le a_1$.

Critical sets of higher order i, $i \ge 1$, defined as $PC_{(i)} = T_2^{i+1}(PC_{-1}$), are important because generally the absorbing areas and the chaotic areas of a non-invertible map are bounded by critical sets.

More general situations and deeper studies of the map T_2 can be obtained and proved if we consider the case : $a_1 = a_2 = a_3 = a$. The Figure 2 presents a chaotic attractor in the space with the parameter $a_1 = a_2 = a_3 = a = 0.99$.
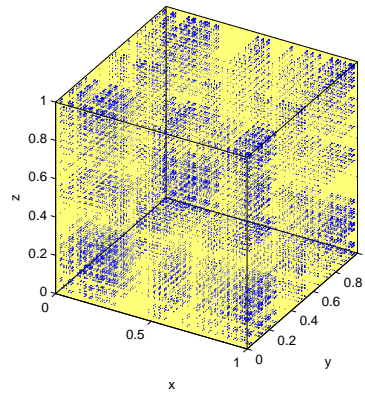
Fig. 2. Chaotic attractor of the map T_2 with the parameter a= 0.99

4   Bifurcation of invariant closed curves

Now we concentrate on presenting the study of T_1 restricting to only 1-parameter $a \in \mathbb{R}_+$. The bifurcation diagram of T_1 in the parameter plane (a, b) is shown in Figure 3, which presents information on stability region.
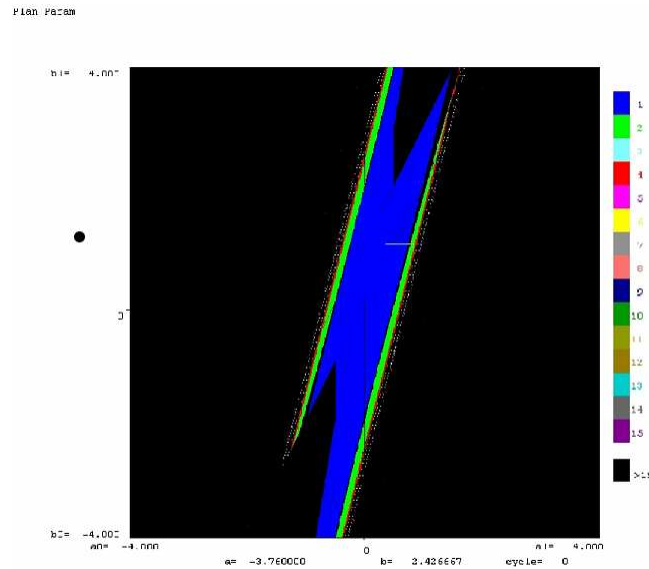

Fig. 3. *Bifurcation diagram for T_1 in the parameter plane (a,b)*

We fix $b = 0.50$ and we vary $a_1 = a_2 = a_3 = a \in \mathbb{R}_+$, so that the map becomes as follows

$$T_3 : \begin{cases} x_{n+1} = 4.a.y_n.(1 - y_n) + 1/2.x_n \\ y_{n+1} = 4.a.z_n.(1 - z_n) + 1/2.y_n \\ z_{n+1} = 4.a.x_n.(1 - x_n) + 1/2.z_n \end{cases} \tag{9}$$

Some algebraic manipulations show that there exists a fixed point x^*, whose coordinates are given by:

$$x^* = \frac{-1 + 8a}{8a}(1, 1, 1)$$

Let us now study the local stability of this fixed point x^*. We have to consider the Jacobian matrix of the map T_3, which is given by

$$J(x, y, z) = \begin{bmatrix} 1/2 & 4.a.(1 - 2y) & 0 \\ 0 & 1/2 & 4a.(1 - 2z) \\ 4.a.(1 - 2x) & 0 & 1/2 \end{bmatrix}$$

and evaluating the Jacobian matrix in x^*, we obtain the matrix

$$J(x^*) = \begin{bmatrix} 1/2 & h & 0 \\ 0 & 1/2 & h \\ h & 0 & 1/2 \end{bmatrix}$$

where

$$h = 4 \cdot a \cdot (1 - \frac{-1 + 8a}{4a})$$

We give a sketch on local stability of the fixed point x^*. So, we can summarize this as follows:

When a belongs to the interval $[0.125, \ 0.41284695471]$, this fixed point is stable.

When $a = 0.41284695471$, a Neimark-Hopf bifurcation appears and an invariant closed curve (ICC) occurs, Figure 4 shows the invariant closed curve(ICC) for the value of parameter $a = 0.4200$. And when a increases, oscillations in the shape of invariant closed curve occur, illustrated in Figure 5. Then a new situation occurs, the curve undergoes a kind of period- dubling. See Figure 6, it is a specific bifurcation to the dimension three. In Figure 7, we see two separated curves but in reality it is impossible to have this case.

Figure 8 represents a new situation related to the creation a loop. A deeper study of this qualitative change from an invariant closed curve in the case of simplest map,

shows that the bifurcation mechanism is more complicated, and it is not directly related to a sudden birth of the weakly chaotic ring.

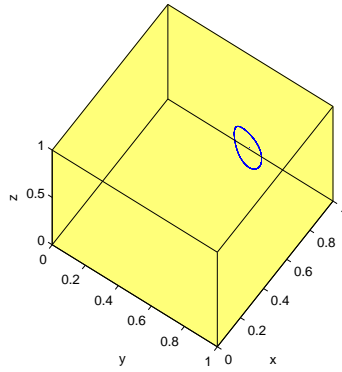For $a = 0.5290$, we obtain a chaotic attractor which is presented in Figure 9, which disappears..
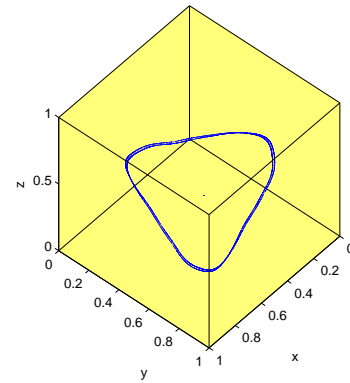

Fig. 4. *a=0.4200:The invariant closed curve of map T3*
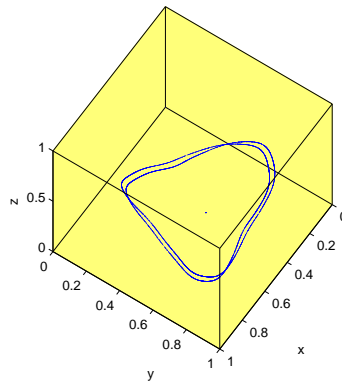
Fig. 5. a= 0.4946: Oscillation in the shape of (ICC)

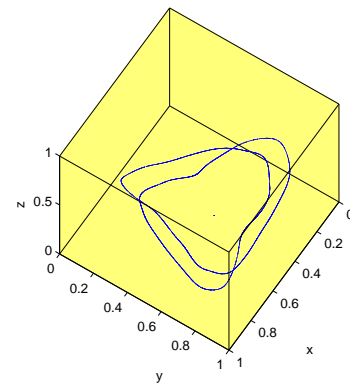Fig. 6. *a=0.4950: Period- dubling of (ICC)*

Fig. 7. a=0.5000: Qualitative change from (ICC)

5   Conclusions

We have studied three-dimensional maps depending on parameters. In Section 2, we have given some properties of T_3, we have studied this three-dimensional system in the parameter plane and in the state space. We have seen that, more one-dimension involves the possibility that new bifurcations occur, and an example was
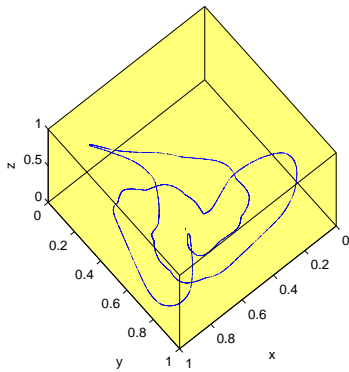
Fig. 8. *a=0.5250: Creation a loop of (ICC)*
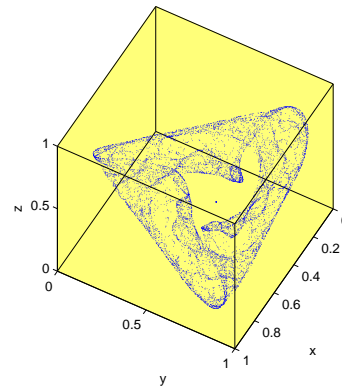
Fig. 9. a=0.5300: Chaotic attractor in the space

given in Section 3. To complete this work, it would be most interesting to introduce a more complete analysis of the global dynamic properties of three-dimensional logistic maps.

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