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Design of Narrow Stopband Recursive Digital Filter

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Abstract: The procedure for design of narrow stopband recursive digital filter realized through parallel connections of two allpass sub-filters is described in this paper. This solution also allows realization of complementary filter, using only one additional adder, and exhibit low sensitivity on coefficients quantization. The method is based on phase approximation of allpass sub-filter. The procedure is very efficient and solution can be obtained within only a few iterations even for large filter order n. Every stopband provides two more equations, one at notch frequency and the other at passband boundary. It is not possible to control attenuation at both passband boundaries, but described procedure provides that achieved attenuations are less or equal to prescribed values. Using this algorithm full control of passband edges is obtained comparing with existing methods where it is not possible.

Keywords: Allpass filters, parallel connection, notch filter, piecewise constant phase characteristic.

1 Introduction

Recursive digital filters realized through parallel connections of two allpass networks have low sensitivity of amplitude characteristic in the passband. Using this approach it is possible to obtain resulting filter to have linear phase characteristic in both, passband and stopband, at the same time. Linear phase in the stopband is not so important characteristic by itself, but ensures that complementary filter also has linear phase in the passband. Existing of these features and a fact that the number of bits used for representation of coefficients of digital filter transfer function depends on the sensitivity of the amplitude characteristics is a reason for increase interest for these filters in the last decade.

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Realization of recursive digital filters through parallel connection of two allpass networks has been described in papers [1], [2] and [3]. The allpass filter design for phase approximation and equalization in Chebyshev sense and some applications are presented in [4] and [5].

The algorithm for synthesis of notch filters using bilinear transformation from analogue domain has been presented in [6]. In this method the first step is design of analogue filter. Notch frequency and stopband width are given. The attenuation is less than 3dB in the passband. The lack of this approach is a fact that width of the stopband is controlled, but at the same time stopband edges ω_{s1} and ω_{s2} remain unknown. For both, analogue prototype and digital filter, notch frequency is not at the center of the stopband and this disagreement is more visible when notch frequency is closer to $\theta = 0$ or $\theta = \pi$. Using design directly in z domain one gets opportunity to have full control of passband edges for every notch frequency. It is possible also to determine transfer function with arbitrary number of notch frequencies using design directly in z domain. In this case input parameters are stopband edges with corresponding attenuations and location of the notch frequencies. Resulting filter will have prescribed or better characteristic comparing with filter obtained using method described in [6].

For these filters the amplitude characteristic directly depends on allpass networks phase difference and it is clear that the design can be achieved just through allpass filters phase approximation. Discontinuity of phase appears during the calculation of digital filters phase which involves certain problems into the design. In this paper complete designing procedure will be described and results will be illustrated on few examples of notch filters design. The proposed procedure for design of narrow stopband filters is very efficient and solution is obtained after only a few iterations. Using this method it is possible completely to control the width of the stopband.

2 Approximation method

Transfer function of recursive digital filter realized using parallel connection of two allpass filters, presented in Figure 1, can be written in the next form

$$F(z) = \frac{1}{2}\left[A_1(z) + A_2(z)\right] \tag{1}$$

where $A_1(z)$ and $A_2(z)$ are allpass filter functions which can be displayed in the following form

$$A_1(z) = z^n \frac{P_n(z^{-1})}{P_n(z)} = z^n \frac{\sum_{i=0}^{n} a_i z^{-i}}{\sum_{i=0}^{n} a_i z^{i}}, \;\; a_0 = 1 \tag{2}$$

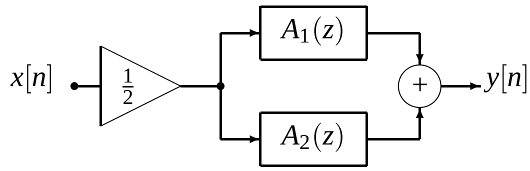


Fig. 1. Realization of IIR selective digital filter using a parallel connection of two allpass filters.

On the unit circle , $z = e^{j\theta}$, functions $A_1(z)$ and $A_2(z)$ are

$$A_i(z) = e^{j\Phi_i(\theta)} \tag{3}$$

and the transfer function (1) can be rewritten in the next form

$$F(e^{j\theta}) = \left|F(e^{j\theta})\right| e^{j\psi(\theta)} \tag{4}$$

where the module of the amplitude characteristic is

$$\left|F(e^{j\theta})\right| = \left|\cos\frac{\Phi_1(\theta) - \Phi_2(\theta)}{2}\right| \tag{5}$$

and the phase is

$$\psi(\theta) = \frac{\Phi_1(\theta) + \Phi_2(\theta)}{2}. \tag{6}$$

The basic idea for this structure is to achieve that in the passband signals in two parallel branches to be in phase. Clearly, the passband and the stopband will be either at frequencies where signals are in phase or contra phase, respectively. It is controlled by the sign of adder inputs. From equation (6) it is obvious that shape of resulting filter phase is directly defined by allpass filter phase. In order to achieve linear phase it is necessary that both allpass phases are linear.

Taking into account the fact that every pair of conjugate-complex poles and corresponding zeros contribute to phase with 2π rad, value of allpass phase at π is $-n\pi$, where n represents allpass filter order. Lowpass or highpass filter could be realized with allpass filters $A_2(z)$ and $A_1(z)$ of order n and $n+1$ respectively, in order to achieve phase difference of π rad which defines the stopband. The easiest way to obtain linear phase is to choose one allpass filter ($A_2(z)$) to be pure delay of order n. With allpass filter $A_1(z)$ of order $n+m$, resulting filter has $m+1$ bands (passbands and stopbands in total) with resulting phase slope of n.

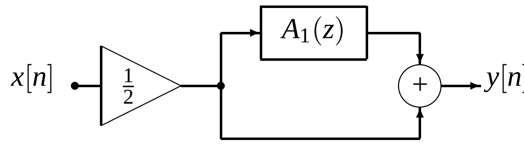


Fig. 2. One realization of notch digital filter.

There is one more way to realize passband (stopband) filter using parallel structure. In this solution both allpass filters are of the same order n, with phase approximating ideal linear phase with slope $n-1$. The stopband is defined by frequencies where phase of allpass filters posses π rad jump. This solution is general in the sense that it is possible to achieve arbitrary phase shape, not only linear, and band edges are defined by location of allpass phase π rad jumps.

Using above explained logic it is possible to realize narrow stopband filters. For this purpose and linear phase case, allpass filters $A_2(z)$ and $A_1(z)$ should be of order n and $n+2$, respectively, with resulting phase of slope n. In one passband phase difference approximates zero and in the other 2π rad. From Figure 3 one can easy conclude that required phase jump of 2π rad in narrow phase transition band is possible to achieve only using double poles. In that region at frequency where phase difference is exactly equal to π rad, notch frequency is positioned. So, to realize filter with m notch frequencies corresponding allpass filters would be of order n and $n+2m$.

In the case when one does not insist on linear phase of resulting filter it is possible to use the simplest structure from Figure 2. The corresponding transfer function $A_1(z)$ can be obtained using very simple procedure. Let

$$A_2(z) = 1 \tag{7}$$

as given in Figure 2, when the equation (5) can be rewritten as

$$\left|F(e^{j\theta})\right| = \left|\cos\frac{\Phi_1(\theta)}{2}\right| \tag{8}$$

i.e.

$$\Psi(\theta) = \frac{\Phi_1(\theta)}{2} \tag{9}$$

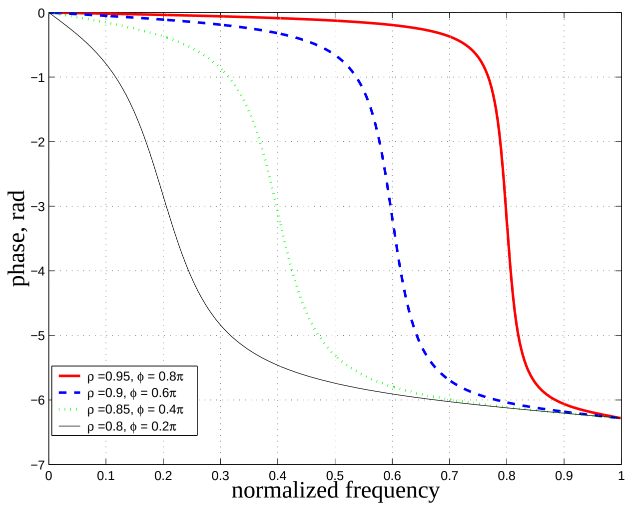


Fig. 3. Phase characteristic of the second order allpass filter for different pole locations.

where allpass filter phase is

$$\Phi_1(\theta, a_i) = n\theta - 2\arctan\frac{\sum_{i=0}^{n} a_i \sin(i\theta)}{\sum_{i=0}^{n} a_i \cos(i\theta)}. \tag{10}$$

Allpass filter coefficients a_i can be obtained easily from poles of transfer function. For the second order section it stands

$$a_1 = -2\rho\cos(\varphi),\ a_2 = \rho^2 \tag{11}$$

where ρ is the module and φ is phase angle of pole of the second order allpass section. The equation 8 is amplitude characteristic of the narrow stopband filter if phase $\Phi_1(\theta, a_i)$ fulfils next conditions

$$\Phi_{1id} = \begin{cases} 0 & ,0 < \theta < \theta_{p1} \\ -\pi & ,\theta = \theta_{g1} \\ -2\pi & ,\theta_{p2} < \theta < \pi \end{cases} \tag{12}$$

where θ_{p1} and θ_{p2} are cutoff frequencies and θ_{g1} is notch frequency, i.e. it is frequency where amplitude characteristic of resulting filter has value exactly equal to zero. These conditions can be fulfilled using allpass filter of the second order $A_1(z)$, in order to realize one notch frequency. Phase characteristic of the second order allpass filter is presented in the Figure 3 and it is approximatively symmetrical (in narrow region) around $\theta = \varphi$. Dependance of frequency θ_{pi}, where phase of the second order allpass filter reachs $-\pi$ rad, from pole's phase angle φ is displayed in

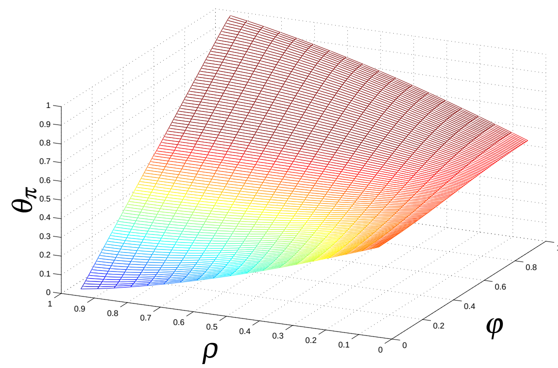


Fig. 4. Frequency θ_π dependance of pole's phase angle φ and pole's module ρ

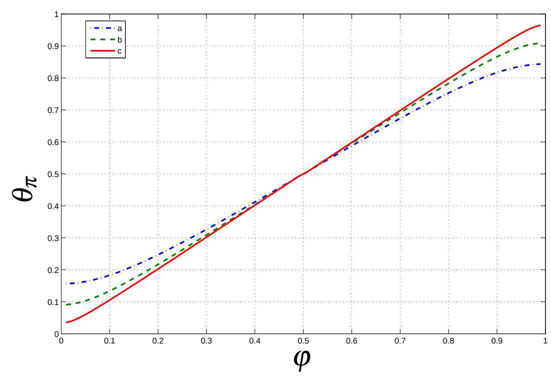


Fig. 5. Frequency θ_π dependance of pole's phase angle φ for different values of pole module ρ=0.75 (a), ρ =0.85 (b) and ρ =0.95 (c).

Figure 5. The phase $\Phi(\theta, a_i)$ approximates ideal phase $\Phi_{1id}(\theta)$, given by equation (12), if the next conditions are satisfied

$$\Phi_1(\theta_{p1}, a_i) = -\varepsilon_1 \qquad \text{for} \qquad \theta_{p1} \leq \frac{\pi}{2} \tag{13a}$$

$$\Phi_1(\theta_{p1}, a_i) = -2\pi + \varepsilon_1 \qquad \text{for} \qquad \theta_{p1} \geq \frac{\pi}{2} \tag{13b}$$

$$\Phi_1(\theta_{g1}, a_i) = -\pi \tag{14}$$

where parameter ε_1 is a phase deviation at frequency θ_{p1}.

It is obvious from Figure 5 that either equation (13a) should be used in the case where notch frequency θ_{g1} is less then $\pi/2$ or equation (13b) if notch frequency θ_{g1} is greater then $\pi/2$. It can be better understood from Figure 6 where

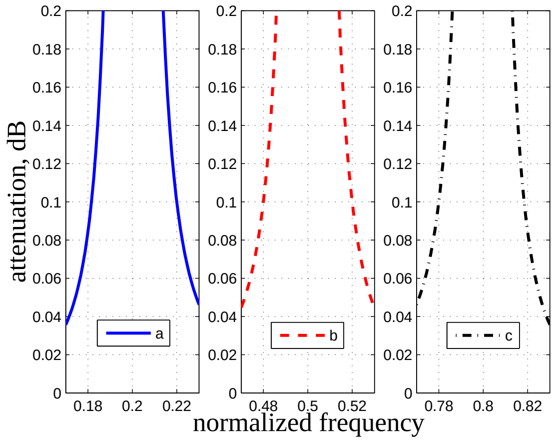


Fig. 6. Attenuation of the second order notch filter with notch frequency at $\theta_g = 0.2\pi$ a), $\theta_g = 0.5\pi$ b) and $\theta_g = 0.8\pi$ c) with maximal attenuation in the passband $A_{max} = 0.1$dB.

attenuations of the second order notch filter for different values of notch frequency θ_{g1} are presented.

The cutoff frequency θ_{p2} is approximatively equal to $\theta_{p2} \approx 2\theta_{g1} - \theta_{p1}$. Substituting of equation (10) in equation (13) the following system of equations is obtained

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}. \tag{15}$$

For the second order filter case, where notch frequency $\theta_{g1} < \pi/2$, above equation can be rewritten in the next form

$$\begin{bmatrix} \frac{\partial \Phi_1(\theta_{p1}, a_i^*)}{\partial a_1} & \frac{\partial \Phi_1(\theta_{p1}, a_i^*)}{\partial a_2} \\ \frac{\partial \Phi_1(\theta_{g1}, a_i^*)}{\partial a_1} & \frac{\partial \Phi_1(\theta_{g1}, a_i^*)}{\partial a_2} \end{bmatrix} \begin{bmatrix} \Delta a_1 \\ \Delta a_2 \end{bmatrix} = \begin{bmatrix} -\varepsilon_1 - \Phi_1(\theta_{p1}, a_i^*) \\ -\pi - \Phi_1(\theta_{g1}, a_i^*) \end{bmatrix} \tag{16}$$

Solving this system of equations increments Δa_1 and Δa_2 can be determined. Initial solutions in every iterative step are marked with a_i^*. New values of coefficients a_i are obtained using expression $a_i = a_i^* + \Delta a_i$, for $i = 1, 2$.

Iterative procedure is ending when the condition is fulfilled that the biggest increment is less than arbitrary small value δ chosen in advance ($\delta = 10^{-10}$ in given examples)

$$|\max(\Delta a_i)| < \delta. \tag{17}$$

Figure 4 gives dependance of frequency θ_π, where the phase of the second order allpass filter reachs value of $-\pi$ rad, on modulo and phase angle of transfer function pole. From Figure 5 it is easy to conclude that better match exists between frequency θ_π and pole phase angle φ for poles with higher Q factor. Based on this fact one can conclude that as good initial solution poles with modulo $\rho^* = 0.9$ and phase angle $\varphi^* = \theta_{g1}$ could be chosen.

Described simple procedure with small modifications can be used for realization of digital filters with more then one notch frequency. Every new notch frequency demands order of filter to rise for two, adding new two equations into system described by (15). In the case of selective digital filter with n stopbands final values for filter coefficients are determined by solving linear system of equation (15), where elements of matrix **A** become

$$\begin{aligned} A_{(2i-1,j)} &= \frac{\partial \Phi_1(\theta_{pi}, a_i^*)}{\partial a_j} \qquad j = 1,2,\ldots,n \\ A_{(2i,j)} &= \frac{\partial \Phi_1(\theta_{gi}, a_i^*)}{\partial a_j} \qquad i = 1,2,\ldots,n/2 \end{aligned} \tag{18}$$

and vector **b** elements are

$$\begin{aligned} b_{(2i-1)} &= (1-i)2\pi - \varepsilon_i - \Phi_1(\theta_{pi}, a_i^*) \\ b_{(2i)} &= (1-2i)\pi - \varepsilon_i - \Phi_1(\theta_{gi}, a_i^*), \qquad i = 1,2,\ldots,n/2. \end{aligned} \tag{19}$$

Now the length of increment vector **x** is n

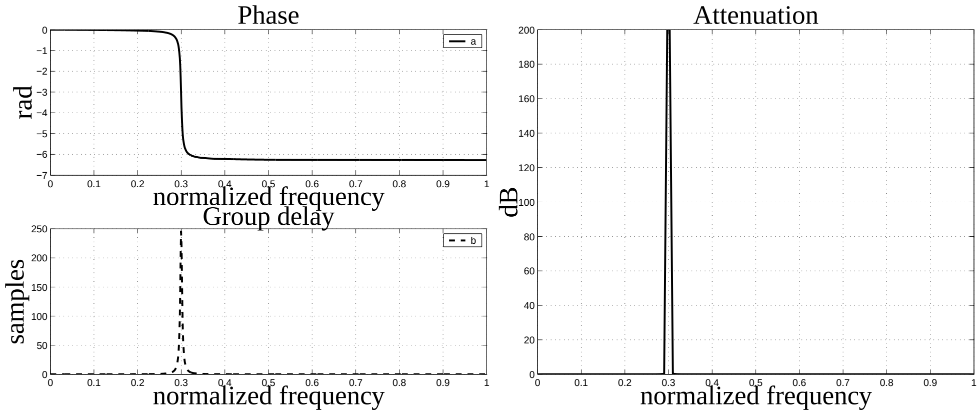
$$\mathbf{x} = [\Delta a_1 \quad \Delta a_2 \quad \ldots \quad \Delta a_n]^T. \tag{20}$$

3 Algorithm

The procedure for determining of coefficients of transfer function of allpass filter $A_1(z)$ can be described with the next algorithm.

1. n stopband central frequencies θ_{gi} and appropriate stopband width θ_{stopi} are given. Also, permitted maximal attenuations A_{max_i} (in dB) at passbands are specified. If notch frequency $\theta_{g1} < \pi/2$ than $\theta_{pi} = \theta_{gi} + \theta_{stopi}/2$ and if notch frequency $\theta_{g1} > \pi/2$ than $\theta_{pi} = \theta_{gi} - \theta_{stopi}/2$. Maximal phase error will be determined as

$$\varepsilon_i = 2\arccos[10^{-A_{max_i}/20}]. \tag{21}$$



(a) (b)

Fig. 7. Phase, group delay and attenuation of the second order notch filter.

As initial solution will be adopted next values for modulo $\rho_i^* = 0.9$ and phase angle $\varphi_i^* = \theta_{gi}$. Poles position chosen by this way allow determining of starting allpass transfer function coefficients a_i^*.

2. Solving system of equations (15) increment of coefficients Δa_i are determined, and using it, new value of filter coefficients a_i can be obtained.

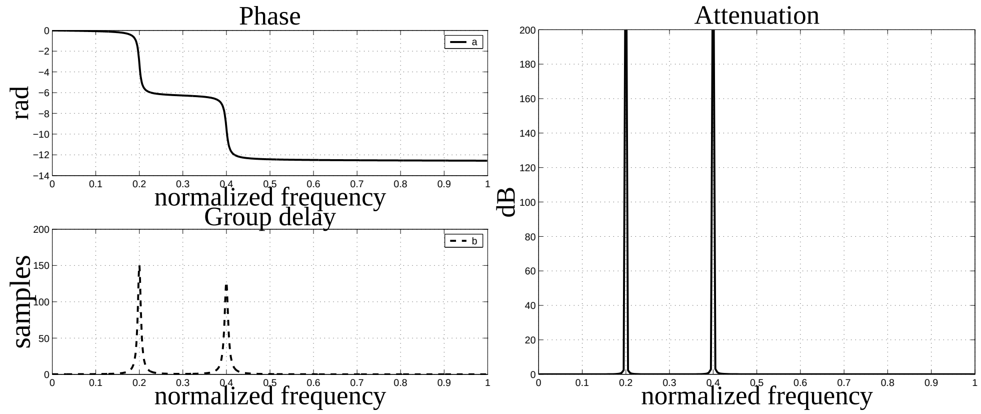
3. Go to step **2.** until condition given by expression (17) is fulfilled , using results from previous iteration as starting values ($a_i^* = a_i$).

4 Examples

The proposed approximation procedure will be illustrated by determining transfer function of bandstop filter with notch frequency $\theta_{g1} = 0.3\pi$, passband border at $\theta_{p1} = 0.25\pi$ with maximal attenuation $A_{max} = 0.01$dB (corresponding phase error has value $\varepsilon_1 = 0.0959$rad). Transfer function coefficients of allpass filter $A_1(z)$ are determined by described algorithm and listed in table 1. Phase and group delay characteristics are given in Figure 7(a) and attenuation in Figure 7(b).

Table 1. Coefficients of the second order allpass transfer function $A_1(z)$, for $\varepsilon_1 = 0.0959$ rad and $A_{max} = 0.01$ dB.

a_0	a_1	a_2
1.00000000000000	−1.18517276238283	1.01633633671555



(a) (b)

Fig. 8. Phase, group delay and attenuation of the fourth order notch filter.

The second example is design of fourth order filter with central stopband frequencies $\omega_{g1} = 0.2\pi$ and $\omega_{g2} = 0.4\pi$ and cut-off frequencies of passband $\omega_{p1} = 0.18\pi$ and $\omega_{p2} = 0.38\pi$ where maximal attenuation is $A_{max1} = A_{max2} = 0.2$dB. It corresponds to phase deviation $\varepsilon_1 = \varepsilon_2 = 0.4275$rad. Coefficients of transfer function of allpass network $A_1(z)$ are listed in table 2.

Table 2. Coefficients of the fourth and the eighth order allpass transfer function $A_1(z)$.

	$n = 4$ $\varepsilon_1 = \varepsilon_2 = 0.42754802731875$ rad $A_{max} = 0.2$ dB	$n = 8$ $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.94140049063259$ rad $A_{max} = 1$ dB
a_0	1.00000000000000	1.00000000000000
a_1	−2.20582606683247	−3.32468650151872
a_2	2.91584992277717	7.19441846153906
a_3	−2.14086635951238	−10.39446001061895
a_4	0.94389994851812	11.63693484350757
a_5		−9.63721614585666
a_6		6.19806616685759
a_7		−2.66461470182447
a_8		0.75554978488909

Amplitude characteristic of resulting filter is displayed in figure 8(b) and phase characteristic and group delay characteristic are presented in figure 8(a).

The last example is design of the eighth order filter with central frequencies of stopband $\omega_{g1} = 0.2\pi$, $\omega_{g2} = 0.3\pi$, $\omega_{g3} = 0.4\pi$ and $\omega_{g4} = 0.5\pi$ and cutoff frequencies of passband $\omega_{p1} = 0.18\pi$, $\omega_{p2} = 0.28\pi$, $\omega_{p3} = 0.38\pi$ and $\omega_{p4} = 0.48\pi$ where maximal attenuations are $A_{max1} = A_{max2} = A_{max3} = A_{max4} = 1$ dB. It corresponds

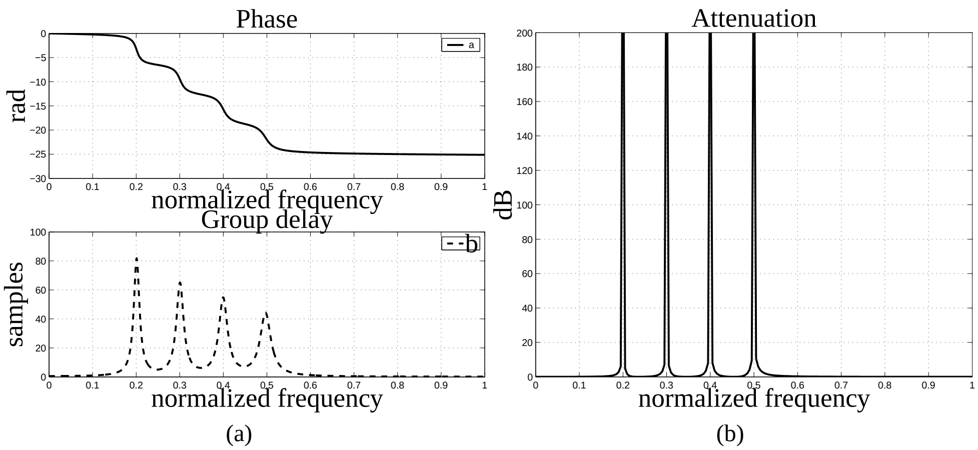


Fig. 9. Phase, group delay and attenuation of the eight order notch filter.

to phase deviations $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.9414$ rad. Coefficients of the transfer function of allpass filter $A_1(z)$ are listed in table 2.

Amplitude characteristic of resulting filter is displayed in figure 9(b) and phase characteristic and group delay characteristic are presented in figure 9(a).

5 Conclusion

The method for design of narrow stopband digital filter realized through parallel connection of two allpas filters is presented in this paper. This procedure is very efficient and final solution can be obtained after only a few iterations. Parallel connections enables realization of complementary filter with only one additional adder and this solution has low sensitivity on coefficients quantization.

The procedure is based on phase approximation of allpass sub-filters. It is possible to realize filter with prescribed number of stopbands, where every new stopband increase for two number of equations in system which must be solved. The maximal attenuation in each passband can be chosen independently, no mater in given examples this attenuation is the same in all passbands.

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