FACTA UNIVERSITATIS (NIŠ)
SERIES: ELECTRONICS AND ENERGETICS VOL. 11, NO.3 (1998), 291-299

NONLINEAR TRANSFORMATION OF ONE–DIMENSIONAL CONSTELLATION POINTS IN ORDER TO ERROR PROBABILITY DECREASING

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Abstract. This paper presents the original gradient method for the determination of the optimal signal constellation points disposition after quantization, when the error probability is minimized under the power constraint. This method consists of the nonlinear transformation and rescaling. The method differs from the method presented in [1], which can be used for uniform constellations with great numbers of points (where the continual approximation can be applied), since it can be applied for generating the nonequiprobable nonuniform constellations containing any number of points.

1. Introduction

The paper[1]is concerned with the performance of nonlinear encoding in a Gaussian noise environment (where the equiprobable constellation points are discussed).It is interesting to compare scaling gain with shaping gain obtained by nonequiprobable signaling on the Gaussian channel. One cost of shaping gain is constellation expansion. The optimal nonlinear encoder is not the one that results in Gaussian distribution at the channel input. One similarity with shaping is that the warped and rescaled constellation (nonlinear transformation constellation) has been expanded. In [1], the optimal nonlinear transformation (warping transformation) is done using the continuous approximation for representing a signal constellation.

The input distribution that achieves the capacity of the Additive White Gaussian Noise (AWGN) channels with an average power constraint is also Gaussian. This paper considers the performance of nonequiprobable nonuniform one–dimensional signal constellations, obtained by shaping and nonlinear transformation when used on the Gaussian channel. In the first part of

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Manuscript received Jule 14, 1998.
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this paper we have considered shaping gain for two interesting quantization procedures of the Gaussian source. The Gaussian source is quantized applying the method given in [3] (the first method), which gives the equiprobable appearance of signal constellation points. After that, the same source is quantized using the Lloyd-Max's iterative quantization method [4] (the second method). In the second more important part of the paper a new gradient method for determining the conditional minimum of the average error probability per symbol under the average power constraint is presented (nonlinear transformation constellation points). Using this method, it is possible to determine the optimal arrangement of constellation points, under the average power constraint, for transmission over the Gaussian channel. This signal constellation transformation method is independent of the quantization method and information source, and adapts the given source signal constellation to the transmission channel.

2. Shaping gain determination for different quantization method

The saving in average power over equiprobable transmission using a one–dimensional constellation is referred to as the shaping gain. The maximal shaping gain is achieved when the Gaussian source is applied. In this section, a Gaussian source quantization, using two different methods of nonlinear quantization, is performed. The analysis is done on the basis of the shaping gain. The Gaussian source is quantized applying the method given in [3], which gives the equiprobable appearance of signal constellation points. After that, the same source is quantized using the Lloyd–Max's iterative quantization method. The numerical analysis is done for the sixteen points of a one–dimensional signal constellation.

Using the first method (I), the decision levels (r_k) and reconstruction levels (m_k) are determined by:

$$\frac{1}{L} = \frac{1}{\sqrt{2\pi P}} \int_{r_k}^{r_{k+1}} \exp -\frac{x^2}{2P} dx \tag{1}$$

$$m_k = \frac{L}{\sqrt{2\pi P}} \int_{r_k}^{r_{k+1}} \exp\left(-\frac{x^2}{2P}\right) dx, \quad k = 1, 2, \ldots, L, \tag{2}$$

where L is a number of levels and P is an average power.

The second method (II) is Max's iterative quantization which leads to r_k and m_k given by [4]:

$$r_{k,opt} = \frac{1}{2}(m_{k,opt}+m_{k-1,opt}); \quad k = 2,3,\dots,L;\ r_{1,opt} = -\infty;\ r_{L+1,opt} = \infty; \tag{3}$$

$$m_{k,opt} = \frac{\int\limits_{r_{k,opt}}^{r_{k+1,opt}} r p_r(r)dr}{\int\limits_{r_{k,opt}}^{r_{k+1,opt}} p_r(r)dr}; \quad k = 1,2,\dots,L \tag{4}$$

where $p_r(r)$ is a probability density function of the source.

The shaping gain (G_S) is defined as difference between SNR (signal to noise ratio) for nonuniform constellation (SNR_n), and SNR for uniform constellation (SNR_u) (similarly as in [5]), when the error probabilities (P_e) are equal and the bit rates are almost equal:

$$G_S(P_e) = SNR_u(P_e) - SNR_n(P_e) \tag{5}$$

The shaping gains for the previous two quantization methods are shown in Fig. 1.

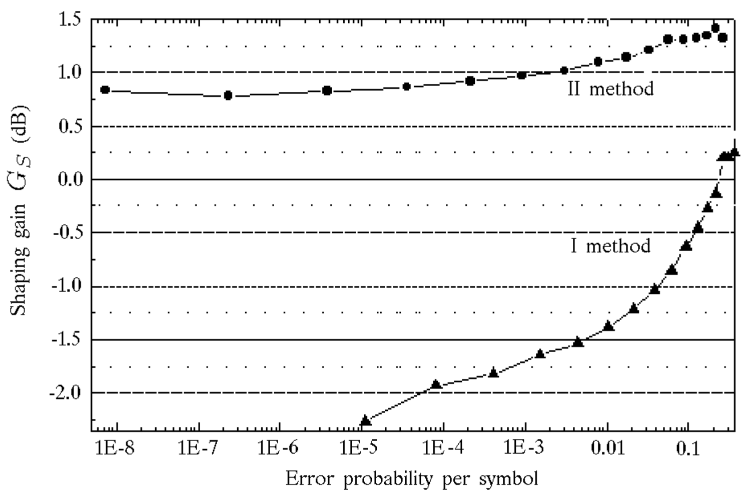

Figure 1. Shaping gain (G_S) for two different quantization methods.

The shaping gain depends very much on the quantization method choice. From Fig. 1. it is evident that the first method has the negative shaping gain value in the wide range of the error probability, so in this case it is better to use the equiprobable transmission with uniform constellation. In the next section we will show how to compensate the negative shaping gain using the appropriate choice of nonlinear transformation and rescaling.

3. Algorithm for determination of optimal signal constellation points disposition

The problem of minimizing the average error probability (P_e) under the average power constraint is the nonlinear programming task where the goal function is P_e and constraint is P (average power). The average error probability and power of one–dimensional signal constellation can be calculated as

$$P_e = 2\sum_{j=1}^{L} \frac{P_j}{\sqrt{2\pi}\sigma_n} \left[\int_{-\infty}^{y_{j,l}} e^{-\frac{r^2}{2\sigma_n^2}} dr + \int_{y_{j,r}}^{+\infty} e^{-\frac{r^2}{2\sigma_n^2}} dr \right], \tag{6}$$

$$P = 2\sum_{j=1}^{L} P_j m_j^2,$$

where σ_n^2 is an average noise power, m_j is the j–th representation level, L is a number of levels

$$P_j = \int_{r_j}^{r_{j+1}} p_r(r)dr,$$

$p_r(r)$ is a probability density function of the Gaussian source, and r_k are the decision levels. y_{jl} and y_{jr} are defined in the following way (see Fig. 2.)

$$\begin{aligned} y_{jl} &= \frac{m_j - m_{j-1}}{2} + \frac{\sigma_n^2}{m_j - m_{j-1}} \ln \frac{P_j}{P_{j-1}}, \quad j = 1, 2, \ldots, L, \\ y_{1l} &= -\infty, \\ y_{jr} &= \frac{m_{j+1} - m_j}{2} + \frac{\sigma_n^2}{m_{j+1} - m_j} \ln \frac{P_j}{P_{j+1}}, \quad j = 1, 2, \ldots, L, \\ y_{Lr} &= +\infty. \end{aligned}$$

In our case the variables in the goal function are mutual dependent, so the solution in closed form using the Lagrange multiplicator method is impossible.The problem is in fact convex programming problem [2], in which

P_e is a convex function and inequality constraint is a convex function (the constraints form a convex set).

Using the gradient method it is necessary to begin from the starting point $M_0(m_1^{(0)}, m_2^{(0)}, \ldots, m_L^{(0)})$ and to go towards the optimum M_i in each iterative step. The next value of m_k is determined by $m_k^{(i+1)} = m_k^{(i)} + \xi_k^{(i)} h^{(i)}$, where $m_k^{(i+1)}$ and $m_k^{(i)}$ represent the k–th representation point at the $i+1$–th and i–th iterative step, respectively. $\xi_k^{(i)}$ is the weight coefficient, and $h^{(i)}$ is a step.

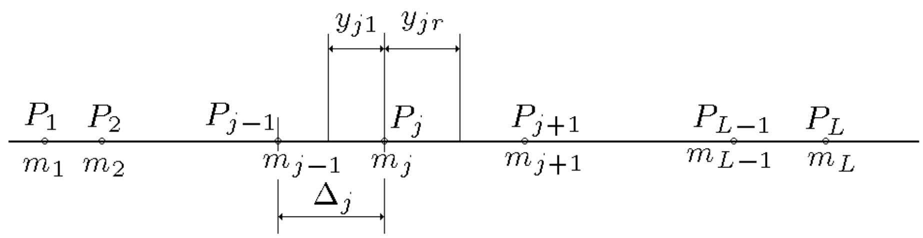

Figure 2. One–dimensional signal constellation and a decision region example.

The algorithm for determination of nonequiprobable nonuniform constellation points disposition is as follows:

0) step: The initialization. The point disposition obtained by described quantization methods application is used as the starting values.

1) step: The average error probability gradient ∇P_e determination

$$\frac{\partial P_e}{\partial \Delta_k} = -\frac{P_k}{\sigma_n\sqrt{2\pi}} e^{-\frac{y_{kl}^2}{2\sigma_n^2}}, \qquad k = 2, \ldots, L;$$

where $\Delta_k = y_{k-1,r} + y_{kl}$.

2) step: Increasing i.e. decreasing distance between two neighbouring points (Δ_k) implies shifting the k–th point to L–th point for some small value $\pm h_1$ in order to keep the relation among remaining points constant

$$m_j^{,(i)} = m_j^{(i)} + h_1,\; m_j^{,,(i)} = m_j^{(i)} - h_1; \quad j = k, k+1, \ldots, L;\; k = 1, 2, \ldots, L.$$

3) step: The calculation of the changed power values due to Δ_k value

variation

$$P_{avk}^{,(i)} = \sum_{j=1}^{L} P_j (m_j^{,(i)})^2,$$

$$P_{avk}^{,,(i)} = \sum_{j=1}^{L} P_j (m_j^{,,(i)})^2, \quad k = 1, 2, \dots, L.$$

4) step: The coefficients $\xi_k^{(i)}$ $(k = 1, 2, \dots, L)$ are chosen so that the distances among the constellation points decrease i.e. increase, proportionally to the speed of error probability changing and the power changing

$$\xi_k^{(i)} = \frac{\frac{\partial P_e}{\partial \Delta_k} \frac{1}{P_{avk}^{\prime(i)} - P_{av}}}{\sum_{j=1}^{N} \frac{\partial P_e}{\partial \Delta_j} \frac{1}{P_{avj}^{\prime(i)} - P_{av}}} + \frac{\frac{P_{avj}^{"(i)} - P_{av}}{\frac{\partial P_e}{\partial \Delta_k}}}{\sum_{j=1}^{N} \frac{|P_{avj}^{"(i)} - P_{av}|}{\frac{\partial P_e}{\partial \Delta_j}}}.$$

5) step: The calculation of new signal point position value $m_k^{(i+1)} = m_k^{(i)} + \xi_k^{(i)} h^{(i)}$, where $h^{(i)}$ is monotonously decreasing function of the ordinal number iterative step.

6) step: The determination of scaling constant, and based of this $m_k^{(i+1)}$ rescaling value

$$\lambda P_{av}^{(i+1)} = P_{av}, \quad m_{k\ rescal}^{(i+1)} = \sqrt{\lambda} m_k^{(i+1)}; \qquad k = 1, 2, \dots, L.$$

7) step: If condition

$$\max |\xi_k^{(i)} \cdot h^{(i)}| \leq \varepsilon,$$

where ε is required to be accurate, is satisfied the procedure is finished or if this condition is not satisfied rescaled values are set as the new signal points position values $m_k^{(i)} = m_{k\ rescal}^{(i+1)}$; $k = 1, 2, \dots, L$ and step 1) is applied.

The nonlinear transformation gain is defined as a difference between SNR before (SNR_{before}) and SNR after (SNR_{after}) the nonlinear transformation (G) of the signal constellation, when the error probabilities (P_e) are equal

$$\begin{aligned} G(P_e) &= SNR_{before}(P_e) - SNR_{after}(P_e) \\ &= 10\log\frac{P_{av}}{\sigma^2} - 10\log\frac{P_{av\ opt}}{\sigma^2_{opt}} \\ &= 10\log\left(\frac{P_{av}}{P_{av\ opt}}\frac{\sigma^2_{opt}}{\sigma^2}\right) \\ &= G_{scaling} + G_n, \end{aligned} \tag{7}$$

where $G_{scaling}$=$10\log(P_{av}/P_{av\ opt})$ is a scaling gain and G_n=$10\log(\sigma^2_{opt}/\sigma^2)$ is a gain due to the nonlinear transformation, which corresponds to the definition given in [1] ($G = G_{scaling} + G_n$).

The error probability decreasing obtained by using the proposed method, for the first quantization method, is illustrated in Fig. 3.

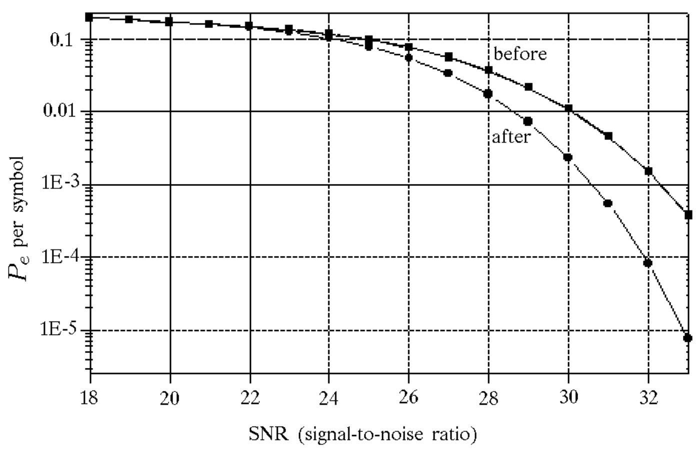

Figure 3. Error probability per symbol before and after nonlinear transformation and rescaling for the first quantization method.

The gain obtained by the nonlinear transformation and rescaling of the signal constellation points initially generated by the first and the second quantization method of the Gaussian source are shown in Fig. 4. The signal constellation design is done separately for each SNR value, and the constellation having the maximal gain is obtained.

The aim of paper [1] is the analysis of nonlinear transformation named warping transformation of the uniform constellation for equiprobable transmission where the solution is obtained using the continual approximation. An increase in nonlinear transformation gain with a decrease of SNR for equiprobable constellations may be observed as in [1]. On the contrary, with nonequiprobable nonuniform constellations, the gain decreases with SNR decreasing. The total gain is equal to the sum of the shaping gain and gain caused by the nonlinear transformation and rescaling.

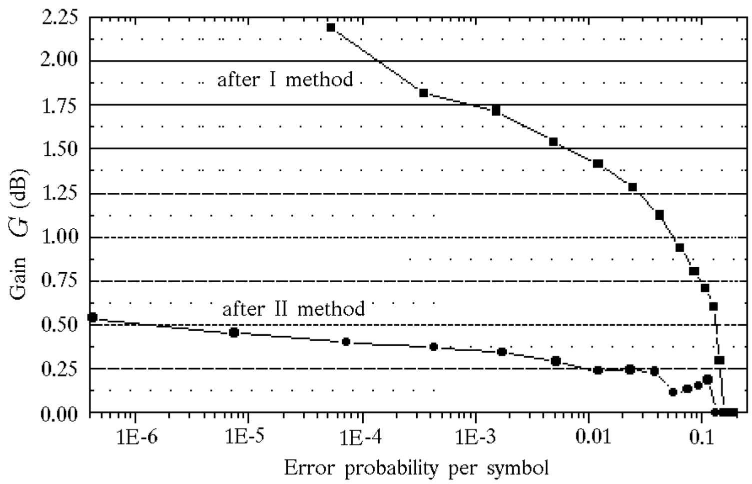

Figure 4. Gain (G) after nonlinear transformation and rescaling.

4. Conclusion

The original gradient method for generating optimal signal constellations minimizes the error probability under the power constraint.The method consists of the nonlinear transformation and rescaling. This signal constellation transformation method is independent of the quantization method and information source, and adapts the given source signal constellation to the transmission channel. Also, we presented a procedure for accurate calculation of shaping gain for any signal constellation.

REFERENCES

1. W. Betts, A.R. Calderbank and Rajiv Laroia: *Performance of Nonuniform Constellations on the Gaussian Channel.* IEEE Transaction on Information Theory, Vol. 40, pp. 1633–1638, September 1994.

2. J.J. Petrić, S. Zlobec: *Nonlinear Programming.* Naučna knjiga, Belgrade, 1983. (in Serbian).

3. F. Sun and H.C.A. Tilborg: *Approaching Capacity by Equiprobable Signaling on the Gaussian Channel.* IEEE Transaction on Information Theory, Vol. 39, pp. 1714–1716, September 1993.

4. N.S. Jayant, P. Noll: *Digital Coding of Waveforms.* Englewood Cliffs, NJ: Prentice–Hall, 1984.

5. F.R. Kschishong, S. Pasupathy: *Optimal Nonuniform Signaling for Gaussion Channels.* IEEE Transaction on Information Theory, Vol. 39, pp. 913–929, May 1993.