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SOME RESULTS ON BINARY DECOMPOSABLE CODES OVER N×M–PSK CONSTELLATIONS

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Abstract. The structure of the encoder and the decoder for binary decomposable codes over $N \times M$–PSK constellations is described. The performance of some codes over the AWGN channel is investigated by means of bounds and simulations. The effect of a differential encoding and a multistage decoding algorithm is studied.

1. Introduction

An $N \times M$–PSK constellation is completely symmetrical and its most natural simply transitive group of isometries is $(Z_M)^N$. An important class of geometrically uniform signal space codes based on this constellation is, thus, identified by the subgroups of $(Z_M)^N$. In this work we deal with a particular kind of subgroups of $(Z_M)^N$ which we call binary decomposable, in analogy with binary decomposable lattices [1].

The latter are widely used in QAM transmission systems for the construction both of efficient multidimensional constellations and good Ungerboeck partition chains for TCM (trellis–coded modulation) codes [2] and can be represented by their code formula:

$$\Lambda = 2^k Z^N + 2^{k-1} c'_{k-1} + \ldots + c'_0 \tag{1}$$

where c'_i $(i = 0, 1, \ldots, k-1)$ is a linear block code over $(Z_2)^N$ and the sum is accomplished in Z^N.

The generic binary decomposable code over an $N \times M$–PSK constellation is represented by a code formula of this kind [3]:

$$S = 2^{r-1} c_{r-1} + \ldots + c_0 \tag{2}$$

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where c_i $(i = 0, 1, \ldots, k-1)$ is still a linear block code over $(Z_2)^N$ but the sum is accomplished in $(Z_M)^N$ with $M = 2^r$.

These PSK codes share with binary decomposable lattices some appealing features. Firstly, it is easy to obtain binary partition chains and to apply the Ungerboeck labeling [1]. Furthermore the parameters can be evaluated from the corresponding parameters of the binary codes involved in the formula. Finally a low–complexity suboptimal multistage decoding (MSD) algorithm can be devised.

In [3] it has also been shown how to impose a natural condition for the rotational invariance and to implement a simple differential encoder.

All of these reveal that such codes are extremely easy to deal with and are particularly appropriate to be employed in TCM schemes.

In this work we describe the structure of the encoder and the decoder for this class of codes, we present the design of some transmission systems and we investigate their performance over the AWGN channel by means of bounds and simulations in the presence of differential encoding and with a MSD algorithm.

2. The code formula

It has been proved [3] that the code formula in Eq. (2) represents a group if and only if the product (logic AND) between each couple of words of c_i is contained in c_{i+1}:

$$c_i \otimes \hat{c}_i \in c_{i+1} \qquad \forall c_i, \hat{c}_i \in c_i \quad (i = 0, \ldots, r-2). \tag{3}$$

The above criterion suggests a way to construct decomposable binary codes over $N \times M$–PSK constellations. Starting from a binary block code c_0, we build $c_0 \otimes c_0$ and choose a code c_1 that contains it as a subcode. The procedure is iterated until the desired depth is reached.

Some performance parameters useful for the code design can be easily evaluated starting from the parameters of the constituent codes.

We denote with (N, k_i, d_i) the parameters of the code c_i and with N_i its error coefficient. Further, we introduce the sets

$$S_i \equiv \{0\} \cup \{2^{r-1} c_{r-1} + \ldots + 2^i c_i | c_i \neq 0\} \qquad (i = 0, \ldots, r-1). \tag{4}$$

It has been shown [3] that the minimum Euclidean distance $d^2_{min}(S)$ of the code S represented by Eq. (2) is given by

$$d^2_{min}(S) = \min_{i=0,\ldots,r-1} [d^2_{min}(S_i)] \tag{5}$$

where

$$d_{min}^2(S_i) = 4R^2 d_i \sin^2 \frac{\pi 2^i}{M} \tag{6}$$

and R is the radius of the M–PSK constellation.

The minimum distance alone allows only a rough estimation of the error probability; a more accurate evaluation requires the knowledge of the error coefficient. A general bound for the error coefficient in terms of the Hamming weights and the number of information bits of the codes has been derived [3].

We first define the set

$$D \equiv \{i | d_{min}(S_i) = d_{min}(S) \quad (i = 0, \dots, r-1)\}. \tag{7}$$

For the sake of notation uniformity we further define $c_r = \{0\}$. Denoting by $L(\hat{c}_i)$ the linear code of all the words having 0 where $\hat{c}_i$ has a 0 and introducing the linear code

$$S(\hat{c}_i) = c_{i+1} \cap L(\hat{c}_i) \tag{8}$$

the error coefficient can be expressed as:

$$N_0(S) = \sum_{i \in D} \sum_{\hat{c}_i} 2^{\dim S(\hat{c}_i)}. \tag{9}$$

$N_0(S)$ can be upperbounded by

$$N_0(S) \leq \sum_{i \in D} N_i 2^{\min(0, K_{i+1} - N - d_i)} \tag{10}$$

and lowerbounded by

$$N_0(S) \geq \sum_{i \in D} N_i 2^{\max(0, K_{i+1} - N + d_i)} \tag{11}$$

It is interesting to observe that the equality is obtained in this lower bound if and only if:

$$c_{i+1}^* \cap L(\hat{c}_i)^* = \{0\} \qquad \forall i \in D. \tag{12}$$

The spectral efficiency of the code S is measured by its rate: since it is clear that each point can be mapped by words of $K = \sum_{i=0}^{r-1} k_i$ bits, the rate is K/N bits/$2D$.

3. Rotational invariance and differential encoding

By virtue of the group properties of S , the rotational invariance is imposed by simply requiring that the sequence of N 1 belongs to S:

$$(1)^N \in S. \tag{13}$$

From Eq. (2) it is obvious that Eq. (13) is satisfied if and only if $(1)^N \in c_0$.

S can be decomposed into a direct sum:

$$S = S_i + S_r \tag{14}$$

where S_r is the subgroup of S generated by $(1)^N$ and S_i is the subgroup of S containing all the sequences whose first element is zero.

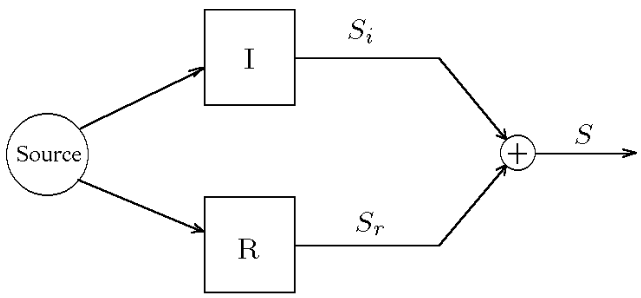

Figure 1. Differential encoder for the group code S.

The transmitter, illustrated in Fig.1, performs a differential encoding only over S_r. The block I contains r systematic binary encoders for the codes c_i. The first input of each encoder is set to zero, so that the whole input is formed by $K - r$ source bits. The outputs of the encoders are combined according to Eq. (2) in order to represent a point of S_i. The remaining r bits are sent to a conventional differential encoder for a single M–PSK constellation. Its output is repeated over N signaling intervals, producing the points of S_r.

This differential encoding scheme may be used both when S is the whole multidimensional constellation and when it is the last level constellation in a Ungerboeck partition chain for TCM.

4. Multistage decoding and receiver structure

Although a maximum likelihood decoding algorithm may be used, its high complexity suggests the search for low complexity suboptimal algorithms.

In particular the decomposition in Eq. (2) suggests a multistage decoding algorithm similar to the one proposed in [5].

We first observe that any codeword identifies an isometry of the Euclidean signal space. If the isometry corresponding to a prescribed codeword $s \in S$ is applied to $r \in R^N$, the transformed vector will be denoted with r^s. If x_0 is the base point assumed for the constellation, every constellation point can be uniquely represented as x_0^s $(s \in S)$.

The algorithm operates as follows. At step 0 the received signal r_0 is decoded in $2(Z_M)^N + c_0$, giving rise to an estimation $\hat{c}_0$ of the projection of the transmitted point over c_0. At step 1 $r_1 \equiv r_0^{-\hat{c}_0}$ is decoded in $2^2(Z_M)^N + 2c_1$ giving rise to $\hat{c}_1$; at step i $r_i \equiv r_{i-1}^{-\hat{c}_{i-1}}$ is decoded in $2^{i+1}(Z_M)^N + 2^i c_i$. Finally, the estimations $\hat{C}_0, \hat{c}_1, \ldots \hat{c}_{r-1}$ are combined according to Eq. (2) in order to recover an estimation of the transmitted point, as illustrated in Fig.2.

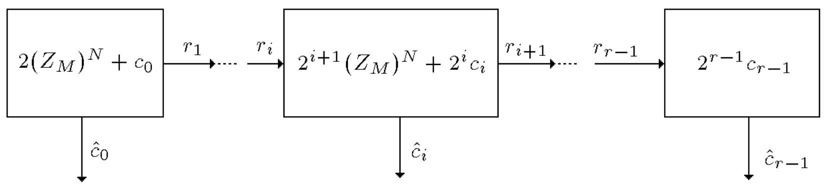

Figure 2. Multistage decoding algorithm.

It is convenient to imagine that block i decodes in $2(Z_{2^{r-i}})^N + c_i$, which is equivalent to $2^{i+1}(Z_M)^N + 2^i c_i$ up to a relabelling of the constellation points. For each coordinate this block chooses the two minimum metrics m_e and m_0 in the subconstellations constituted by even and odd points respectively and it passes them to a soft decoder for the binary code c_i, which determines the word $\hat{c}_i$.

It has been proved [3] that such MSD algorithm is bounded–distance as in the case of lattices, i.e. it behaves as the maximum likelihood decoding when the distance between the received signal and a point of S is smaller than $d_{min}(S)/2$.

We notice that the algorithm is still bounded distance even if the soft decoding algorithms for the binary codes are not optimal but bounded distance themselves. Therefore, if necessary, the decoding complexity may be further reduced, using simpler algorithms for the codes.

The practical relevance of this MSD algorithm stems from its lower complexity in comparison with the optimal algorithm and from its good performance, close to the optimum in the typical range of SNR's. Indeed the dominant error exponent is the same as for the maximum likelihood algorithm and the impairment in performance can be measured in a first approximation by means of the equivalent error coefficient of the bounded distance algorithm.

Since each block in the MSD algorithm works independently from the others, the equivalent error coefficient is the sum of the error coefficients of the codes $2(Z_{2^{r-i}})^M + c_i \ \forall i \in D$:

$$N_{0,BD}(S) = \sum_{i \in D} N_i P_i \tag{15}$$

where $P_i = 2^{d_i}$ if $i < r-1$ and $P_{r-1} = 1$.

The output of the multistage decoder is further processed by a differential decoder and finally delivered to an inverse mapper which recovers the information bits, through a projection of the estimated point over the component codes.

5. Performance of the codes

In Table 1 we report some rotationally invariant binary decomposable codes with their code formula and their main parameters; it is worth noting that all these codes can be obtained through the squaring construction so that they can also be decoded by means of the optimal algorithm detailed in [4]. For the codes #7, #9, #10 the MSD algorithm coincides with the maximum likelihood decoding algorithm. For the other codes, instead, this algorithm is suboptimal, even when its error coefficient is equal to the error coefficient of the code.

In Fig.3 we illustrate the performance of the codes in Table 1 over the AWGN channel in the case of optimal decoding. The codes are compared at fixed error probability; as a reference the uncoded PSK modulations are also indicated. All the points have been evaluated through the union bound, on the basis of a significant portion of the distance spectrum. E_b denotes the average energy per bit of information, while $N_0/2$ is the power spectral density of the noise.

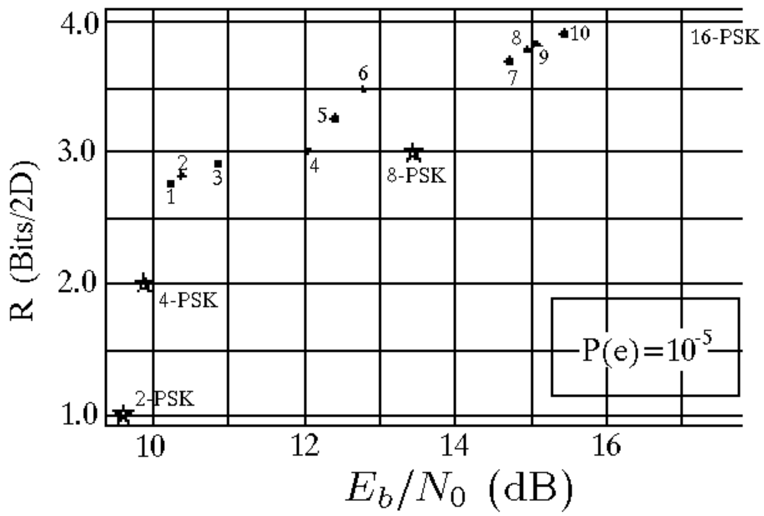

Figure 3. (SNR Efficiency)/Rate Trade–off.

In Fig. 4 we show the decrease of the performance in the case of the code #6 due to the use of the differential encoding and the MSD algorithm with optimal soft decoder at each stage. In the diagram we compare the union bound of the point error probability for the case of optimal decoding with the point error probability and the bit error probability for the case of MSD and differential encoding. These latter two results have been obtained by means of simulations on the computer and are guaranteed to differ from the correct values by less than 20% with probability of 0.8.

It is interesting to observe that the simulation values of the point error probability are very near to the union bound results, which means that the decrease of performance due to the MSD algorithm is extremely small. That is motivated by the fact that $N_0 = N_{0,BD}$ as it is illustrated in Table 1.

Obviously, the crossover between the first two curves depends exclusively on the fact the union bound is a loose upper bound for low SNR's.

In Table 2 we report the decoding complexity per bit of information, calculated as the number of sums and comparisons between real numbers, for the cases of the optimal decoding algorithm detailed in [4] and the MSD algorithm described in section 4.

Table 1. Performance of some binary decomposable codes over $N \times M$–PSK constellations

#	Code formula	$d_{min}(S)/R$	N_0	$N_{0,BD}$	Bits/$2D$
1	$4(Z_{16})^{16} + 2(16,11,4) + (16,1,16)$	1.414	32	32	2.75
2	$8(Z_{16})^{32} + 4(32,31,2) + 2(32,26,4) + (32,1,32)$	1.531	9920	19840	2.81
3	$4(Z_{16})^{32} + 2(64,57,4) + (64,1,64)$	1.414	128	128	2.91
4	$4(Z_{16})^{8} + 2(8,7,2) + (8,1,8)$	1.082	112	112	3.00
5	$4(Z_{16})^{16} + 2(16,15,2) + (16,5,8)$	1.082	480	480	3.25
6	$4(Z_{16})^{32} + 2(32,31,2) + (32,16,8)$	1.082	1984	1984	3.47
7	$2(Z_{16})^{16} + (16,11,4)$	0.765	32	32	3.69
8	$4(Z_{16})^{32} + 2(32,31,2) + (32,26,4)$	0.780	9920	19840	3.78
9	$2(Z_{16})^{32} + (32,26,4)$	0.765	64	64	3.81
10	$2(Z_{16})^{64} + (64,57,4)$	0.765	128	128	3.89

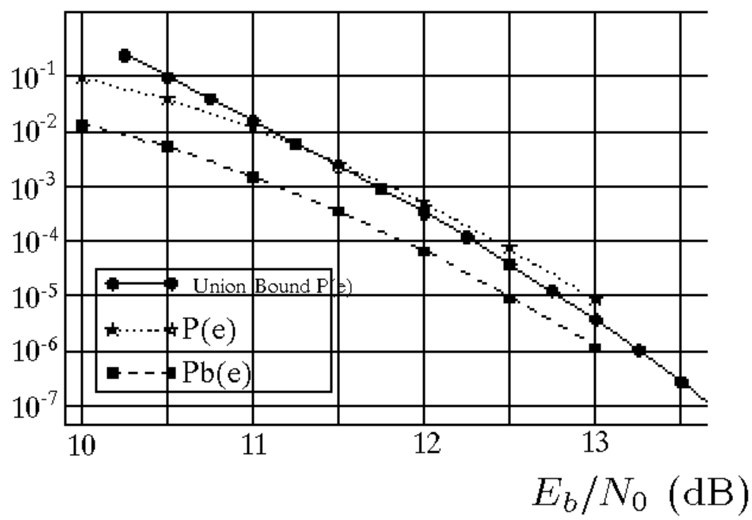

Figure 4. Performance of code #6.

Table 2. Decoding complexity per bit of information
O: optimal decoding algorithm; MS: multistage algorithm

#	1	2	3	4	5	6	7	8	9	10
O	11.6	117	57.8	3.29	12.3	116	4.32	43.4	9.96	21.6
MS	6.52	14.7	29.4	1.17	3.21	27.8	4.32	10.7	9.96	21.6

6. Conclusions

In this work we summarized the definition and the main properties of binary decomposable codes over $N \times M$–PSK constellations. We detailed the implementation of the encoder and the decoder in the case of the application of a differential encoding and a MSD algorithm. We presented some binary decomposable codes and we investigated their performance on the AWGN channel. We showed that the impairment in the performance due to the suboptimal decoding algorithm can be neglected, while the decrease of the decoding complexity is often dramatic.

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