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OPTICAL SIGNALS IN COLORED GAUSSIAN NOISE

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Abstract. In this paper, detection of ASK optical signal in colored Gaussian noise is performed. The detection of signals in colored noise is discussed because the assumption that white noise is present in real systems is often invalid. Beside the colored noise, the laser phase noise influence on the system performance is considered. For these conditions, the likelihood ratio and the optimal receiver are determined and the system error probability is calculated.

1. Introduction

In this paper detection of Amplitude Shift Keying ASK optical signals in colored Gaussian noise is considered:

$$r(t) = a_i A \cos(\omega_0 t + \theta) + n(t), \qquad 0 \le t \le T. \tag{1}$$

The multiplicative factor a_i has values 0 and 1 for hypothesis H_0 and for hypothesis H_1, respectively. Because of the laser phase noise, the optical signal phase θ has Gaussian distribution [1]:

$$p(\theta) = \frac{1}{\sigma_\theta \sqrt{2\pi}} \exp\left(-\frac{\theta^2}{2\sigma_\theta^2}\right), \qquad \theta \in [-\pi, \pi]. \tag{2}$$

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It is assumed that the autocorrelation function $R_n(\tau)$ of colored Gaussian noise is a usual exponential function:

$$R_n(\tau) = \frac{1}{4} N_0 \omega_0 \exp[-\omega_0 \mid \tau \mid]. \tag{3}$$

2. Likelihood ratio

Beside a Fourier series expansion in terms of sine and cosine function with appropriate weighting coefficients, there is a Karhunen–Loeve expansion [2]:

$$r(t) = \sum_{k=1}^{+\infty} r_k f_k(t). \tag{4}$$

At Karhunen–Loeve expansions, signal is present by set of orthonormal functions $f_k(t)$, defined over the interval [0,T] and uncorrelated weighting coefficients r_k. Orthonormal functions are solutions of a homogeneous integral equation:

$$\int_0^T f_k(t_2) R_n(t_1 - t_2) dt_2 = \lambda_k f_k(t_1) \tag{5}$$

where λ_k is an eigenvalue. The coefficients r_k may be generated by inserting $r(t)$ into a linear filter matched to $f_k(t)$ and sampling the output at time $t = T$:

$$r_k = \int_0^T r(t) f_k(t) dt. \tag{6}$$

If the $s(t)$ is known signal in colored Gaussian noise:

$$r(t) = s(t) + n(t), \tag{7}$$

the coefficients r_k result from linear operation on the Gaussian process $r(t)$. They are therefore Gaussian with the mean value s_k and variance λ_k. The s_k is a Karhunen–Loeve coefficient of signal $s(t)$. Because the coefficients r_k are uncorrelated, they are also statistically independent and their joint density function is:

$$p(r) = \lim_{N \to +\infty} \prod_{k=1}^{N} p(r_k) = C \exp\left[-\sum_{k=1}^{+\infty} \frac{(r_k - s_k)^2}{2\lambda_k}\right]. \tag{8}$$

If $s(t)$ is optical ASK signal with unknown phase θ (Eq. (1)), than the conditional likelihood functions for hypothesis H_0 and H_1 can be written as:

$$p_i(r/\theta) = C \exp\left[-\sum_{k=1}^{+\infty} \frac{(r_k - s_{ik})^2}{2\lambda_k}\right], \tag{9}$$

where $i = 0$ for hypothesis H_0 and $i = 1$ for hypothesis H_1. If

$$\sum_{k=1}^{+\infty} \frac{s_{ik} f_k(t)}{\lambda_k} \tag{10}$$

denotes with $h_i(t,\theta)$, the conditional likelihood functions become:

$$\begin{aligned} p_0(r) &= C \exp\left[-\sum_{k=1}^{+\infty} \frac{r_k^2}{2\lambda_k}\right] \\ p_1(r/\theta) &= C \exp\left[-\sum_{k=1}^{+\infty} \frac{r_k^2}{2\lambda_k}\right] \exp\left\{\int_0^T \left[r(t,\theta) - \frac{1}{2} s_1(t,\theta)\right] h_1(t,\theta) dt\right\}. \end{aligned} \tag{11}$$

The $h_1(t,\theta)$ is a solution of the Fredholm integral of the first kind:

$$\int_0^T R_n(t-\tau) h_1(\tau,\theta) d\tau = s_1(t,\theta). \tag{12}$$

For autocorrelation function $R_n(\tau)$ defined by equation (3), solution of integral equation (12) is [2]:

$$\begin{aligned} h_1(t,\theta) = \frac{2}{N_0 \omega_0^2} &\{[\omega_0 s_1(0,\theta) - s_1'(0,\theta)]\delta(t) \\ &+ [\omega_0 s_1(T,\theta) + s_1'(T,\theta)]\delta(t-T) + \omega_0^2 s_1(t,\theta) - s_1''(t,\theta)\}, \\ &\qquad 0 \le t \le T. \end{aligned} \tag{13}$$

Now, the likelihood ratio after some manipulations is seen to be:

$$\begin{aligned} \lambda(r) = \frac{p_1(r)}{p_0(r)} &= \frac{\int_{-\pi}^{\pi} p_1(r/\theta) p(\theta) d\theta}{p_0(r)} \\ &= \exp(-g) \int_{-\pi}^{\pi} \exp[p\cos(\theta + \phi) - d\cos 2\theta] p(\theta) d\theta \end{aligned} \tag{14}$$

where

$$\begin{aligned}
p &= \frac{2A^2T}{N_0}\sqrt{\left[\frac{r(T)+r(0)}{AT\omega_0}+2q\cos\theta_0\right]^2+\left[\frac{r(T)-r(0)}{AT\omega_0}+2q\sin\theta_0\right]^2} \\
\phi &= \arctan\frac{\dfrac{r(T)-r(0)}{AT\omega_0}+2q\sin\theta_0}{\dfrac{r(T)+r(0)}{AT\omega_0}+2q\cos\theta_0} \\
d &= \frac{A^2}{N_0\omega_0} \\
g &= \frac{A^2}{N_0\omega_0}+\frac{A^2T}{N_0} \\
q &= \sqrt{x^2+y^2} \\
\theta_0 &= \arctan\frac{y}{x} \\
x &= \frac{1}{AT}\int_0^T r(t)\cos\omega_0 t dt \\
y &= \frac{1}{AT}\int_0^T r(t)\sin\omega_0 t dt
\end{aligned} \tag{15}$$

Integral (14) can be solved only by expansion of exponential terms into series of modified Bessel function of the first kind:

$$\begin{aligned}
\exp[p\cos(\theta+\phi)] &= \sum_{n=-\infty}^{+\infty} I_n(p)\cos n(\theta+\phi) \\
\exp[-d\cos 2\theta] &= \sum_{m=-\infty}^{+\infty} I_m(-d)\cos 2m\theta
\end{aligned} \tag{16}$$

and expressing Gaussian probability density in the form of a Fourier cosine series:

$$p(\theta) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty}\exp\left(-\frac{k^2\sigma_\theta^2}{2}\right)\cos k\theta \qquad \theta\in[-\pi,\pi]. \tag{17}$$

After this, the final form of likelihood ratio is:

$$\begin{aligned}\lambda(r) = \exp(-g)\Big\{ & I_0(p) I_0(-d) + 2I_0(-d)\sum_{n=1}^{+\infty} I_n(p)\exp\Big(-\frac{n^2\sigma_\phi^2}{2}\Big)\cos n\phi \\ & + 2I_0(p)\sum_{n=1}^{+\infty} I_n(-d)\exp(-2n^2\sigma_\phi^2) \\ & + 2\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} I_n(p) I_m(-d)\exp\Big(-\frac{(n+2m)^2\sigma_\phi^2}{2}\Big)\cos n\phi \\ & + 2\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty} I_n(p) I_m(-d)\exp\Big(-\frac{(n-2m)^2\sigma_\phi^2}{2}\Big)\cos n\phi\Big\}.\end{aligned} \tag{18}$$

The decision rule is to choose H_1 if $\lambda(r) > \lambda_0$. The above equation (18) specifies the optimal receiver construction procedure. For the received signal $r(t)$, the receiver calculates the likelihood ratio $\lambda(r)$, compares it with λ_0 and makes decision about which signal is sent by transmitter.

3. Performance of optimal receiver

Since the noise $n(t)$ is Gaussian process, the conditional variables x and y (see Eq. (15)) also have Gaussian distribution with mean values:

$$\overline{x/\theta} = 0 \qquad \overline{y/\theta} = 0 \tag{19}$$

for hypothesis H_0 and

$$\overline{x/\theta} = \frac{\cos\theta}{2} \qquad \overline{y/\theta} = -\frac{\sin\theta}{2} \tag{20}$$

for hypothesis H_1. The variances are same for both hypothesis and they are

$$\sigma_x^2 = \frac{N_0}{8A^2T}\left[1 - \frac{1-\exp(-\omega_0 T)}{\omega_0 T}\right] \tag{21}$$

for conditional variable x and

$$\sigma_y^2 = \frac{N_0}{8A^2T}\left[1 + \frac{1-\exp(-\omega_0 T)}{\omega_0 T}\right] \tag{22}$$

for conditional variable y. Because the correlation coefficient of conditional variables x and y is zero for both hypothesis, the conditional variables x and

y are statistically independent and their conditional joint density functions are

$$p_0(x,y/\theta) = p_0(x/\theta)p_0(y/\theta) = p_0(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)$$

$$p_1(x,y/\theta) = p_0(x/\theta)p_0(y/\theta) = \frac{1}{2\pi\sigma_x\sigma_y}\exp\left(-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right)\exp\left(-\frac{\sigma_x^2+\sigma_y^2}{16\sigma_x^2\sigma_y^2}\right)$$
$$\exp\left[-\frac{1}{2}\sqrt{\frac{x^2}{\sigma_x^4}+\frac{y^2}{\sigma_y^4}}\cos\left(\theta+\arctan\frac{y\sigma_x^2}{x\sigma_y^2}\right)-\frac{\sigma_y^2-\sigma_x^2}{16\sigma_x^2\sigma_y^2}\cos 2\theta\right]. \tag{23}$$

Comparing equations (23) with likelihood ratio term (14), the variables p, ϕ, d, g and the joint probability $p_1(x,y)$ are seen to be:

$$\begin{aligned} p &= \frac{1}{2}\sqrt{\frac{x^2}{\sigma_x^4}+\frac{y^2}{\sigma_y^4}} \\ \phi &= \arctan\frac{y\sigma_x^2}{x\sigma_y^2} \\ d &= \frac{\sigma_y^2-\sigma_x^2}{16\sigma_x^2\sigma_y^2} \\ g &= \frac{\sigma_y^2+\sigma_x^2}{16\sigma_x^2\sigma_y^2} \\ p_1(x,y) &= p_0(x,y)\lambda(r). \end{aligned} \tag{24}$$

Since the new equations for variables p, ϕ, d and q are determined, the optimal receiver is designed in block diagram form and shown in Figure 1. The optimal receiver calculates random variables x and y (Eq. (15)), afterwards it determines p and ϕ and at last, the likelihood ratio $\lambda(r)$, compares it with λ_0 and makes decision about which signal is sent by transmitter.

It is possible to calculate the system error probability for determined joint densities $p_0(x,y)$ and $p_1(x,y)$

$$P_e = P(H_0)\int\int_{D_1} p_0(x,y)dxdy + P(H_1)\int\int_{D_0} p_1(x,y)dxdy. \tag{25}$$

Decision areas D_0 and D_1 are determined from the likelihood ratio term. The area D_0 in plane $x0y$ is determined from the condition that $\lambda(x,y) < 1$ and the area D_1 from $\lambda(x,y) > 1$. In case when a priori probabilities are equal, the error probabilities have been evaluated for the optimal receiver suggested in this work for detection of optical signals in colored Gaussian noise and are shown in Figure 2. The Figure 2. indicates that the system error probability decreases with increasing the signal to noise ratio and with reducing the quantum noise variance.

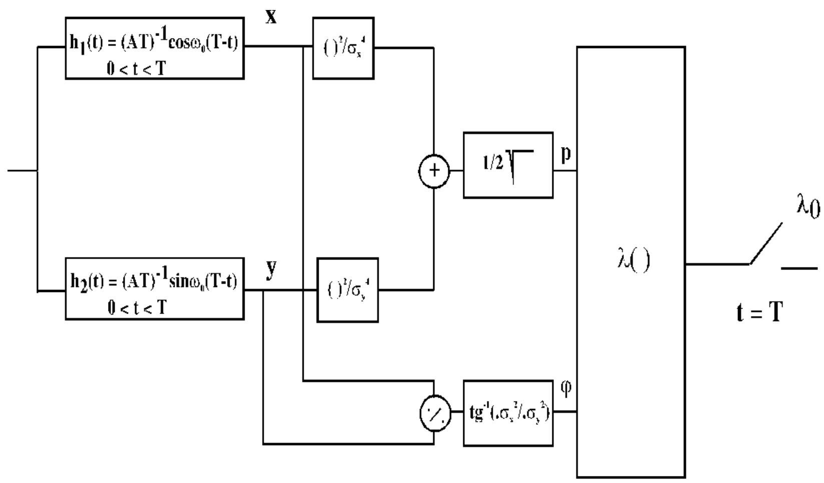

Figure 1. Optimal receiver.

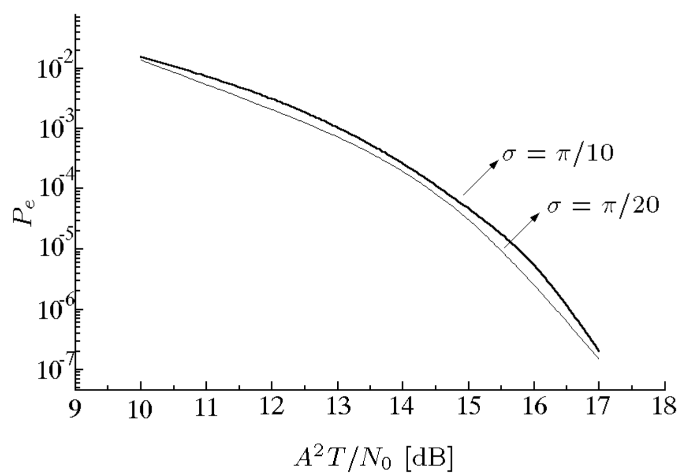

Figure 2. Error probability dependence on ratio A^2T/N_0 in dB

4. Conclusion

In this paper, the optical communication system with ASK modulation is considered. Detection of optical signal in colored Gaussian noise, when there is also the laser phase noise, is performed. The optimal optical receiver is designed in block diagram form, an expression for the error probability is derived and the error probability dependence on signal to noise ratio for different values of quantum noise variance is graphically shown.

It is known in the literature that the system with the optimal receiver achieve smaller error probability for the same signal power. On the other side, these systems require the complex numerical simulation.

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