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THE QUANTIZATION NOISE GIVEN BY PHASE CORRECTORS IN DIGITAL RECEIVERS

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Abstract. The paper analysis probability density of the quantization noise given by the sine and cosine functions of the phase θ, when the phase θ is uniform distibuted in the interval $[0, 2\pi]$. By the Monte Carlo simulation, the phase error given by the quantization of $\sin\theta$ and $\cos\theta$ is determined and compared with the error obtained by unifirm quantization of the phase.

1. Introduction

Phase rotators and freqiency translators are devices needed within full digital modems for the purpose of correcting phase and frequency offsets of the received signal's carrier. Those correctors must be driven by devices that provides values for the desired phase or frequency corrections.

The digital phase and frequency correctors usually employs the ortogonal components $\sin\theta$ and $\cos\theta$, rather then the instantaneous phase θ itself, which must be converted. The phase measurement and filtering elements and the numerical values of θ will be stored in an accumulator or a register. Tipical estimators are likely to quantize phase into fine increments.

The sine and cosine tables are stored in a ROM and the quantized values of θ serves as an adress. For digital implementations, the problem is to find the relation between the number of adress lines and the output word length for each value of $\sin\theta$ and $\cos\theta$, so that the memory size not to be excessively uncreased and, in the same time, the precision to be good enough so that the degradation due to quantization to be neglected.

In [1] such an analysis is developed, using the assumption that the quantization errors are uniform distributed simultaneously for $\sin\theta$ and $\cos\theta$, but it is difficult to accept this hypothesis.

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2. The pdf of the phase quantization error

When no information is possesed about the phase distribution, this can be considered uniform distributed within the interval $[0, 2\pi]$. Therefore the probability density function is

$$p_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & \text{for } \theta \in [0, 2\pi) \\ 0 & \text{in rest} \end{cases} \tag{1}$$

If the phase is quantized with b bits, the circle is divided into 2^b increments and the quantization error is uniform distributed within an increment, so the r.m.s. quantization phase noise is

$$\sigma_{\Delta\theta} = \frac{\Delta}{\sqrt{12}} \tag{2}$$

where

$$\begin{aligned} \Delta\theta &= Q(\theta) - \theta = 2\pi k 2^{-b} - \theta \text{ rad} \\ &= k2^{-b} - \theta \text{ complete rotations} \\ &\text{for } \theta \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right) \text{ and } k = 0, \dots, 2^b - 1 \end{aligned} \tag{3}$$

and

$$\Delta = 2\pi 2^{-b} \text{ rad } = 2^{-b} \text{ complete rotations} \tag{4}$$

The probability density function (pdf) of $\sin\theta$ and $\cos\theta$ is no more uniform, because sine and cosine are not linear functions. Due to [2] for those two functions, the pdf is given by

$$p_s(x) = p_c(x) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{1-x^2}} & \text{for } x \in [-1, 1] \\ 0 & \text{in rest} \end{cases} \tag{5}$$

Considering that b bits are used for uniform, simmetric quantization of sine and cosine functions, the quantized value are

$$Q(\sin\theta) = \frac{2k}{2^b - 1}; \qquad \sin\theta \in \left[\frac{2k-1}{2^b-1}, \frac{2k+1}{2^b-1}\right] \tag{6}$$

respectively

$$Q(\cos\theta) = \frac{2k}{2^b - 1}; \qquad \cos\theta \in \left[\frac{2k-1}{2^b-1}, \frac{2k+1}{2^b-1}\right] \tag{7}$$

where

$$k \in Z, \quad k \in \left\{-2(2^{b-1}-1), \dots, (2^{b-1}-1)\right\} \tag{8}$$

and $Q(\cdot)$ is the uniform quantization function.

The quantization error

$$\begin{aligned} \Delta s &= Q(\sin\theta) - \sin\theta \\ \Delta c &= Q(\cos\theta) - \cos\theta \end{aligned} \tag{9}$$

has the same pdf for both functions, given by

$$\begin{aligned} p_{\Delta s}(x) &= p_{\Delta c}(x) \\ &= \frac{1}{\pi} \sum_{k=-(2^{b-1}-1)}^{2^{b-1}-1} \frac{1}{\sqrt{1-\left(\dfrac{2k}{2^b-1}-x\right)^2}} \\ &\text{for } x \in \left(-\frac{1}{2^b-1}, \frac{1}{2^b-1}\right) \end{aligned} \tag{10}$$

This function is reprezented in Figure 1. for 8–bit quantization. It's easy to see that this error is far to be uniform in the conditions given by (1).

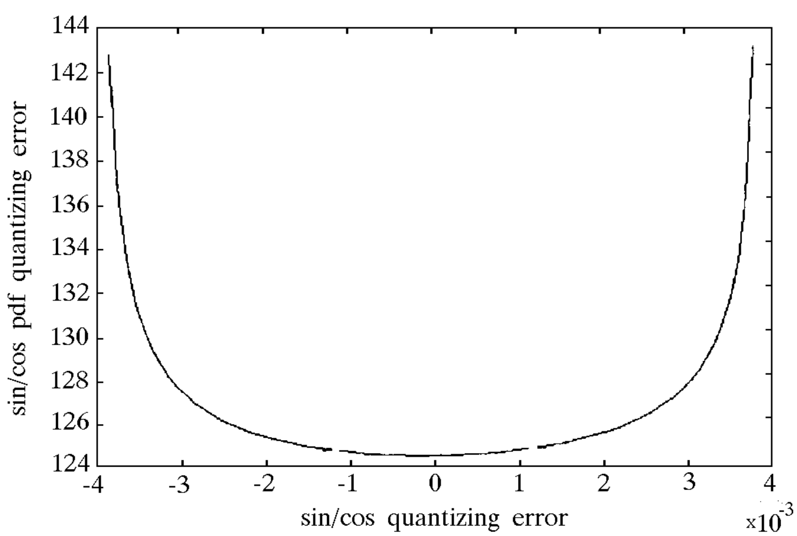

Figure 1. The pdf of the quantization error for $\sin\theta$ and $\cos\theta$.

3. The indirect phase quantization error

Under the assumption that the phase error is evaluated using the relation

$$Q_1(\theta) = \arctan\left[\frac{Q(\sin\theta)}{Q(\cos\theta)}\right] \tag{11}$$

which define the "indirect quantization of the phase", the quantization error is

$$\begin{aligned}\Delta\theta &= \arctan\left[\frac{Q(\sin\theta)}{Q(\cos\theta)}\right] - \theta \\ &= \arctan\left(\frac{\sin\theta + \Delta s}{\cos\theta + \Delta c}\right) - \theta\end{aligned} \tag{12}$$

Because the quantization errors distribution for sine and cosine functions is not uniform and those functions are not independent, the rms quantizing phase error is evaluated using Monte Carlo simulation [3]. Using 16 sets of 1000 trials of uniform distributed θ values represented by 4 to 32 bits, the ratio between the rms error for direct quantization is calculated and represented in Figure 2. The mean of those results for those 16 sets is depicted in Figure 3. and the variance in Figure 4.

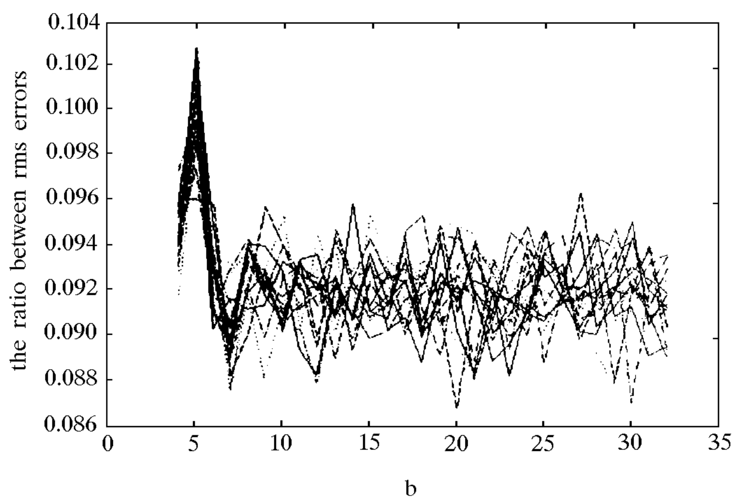

Figure 2. The ratio between rms quantization phase errors for indirect and direct quantization.

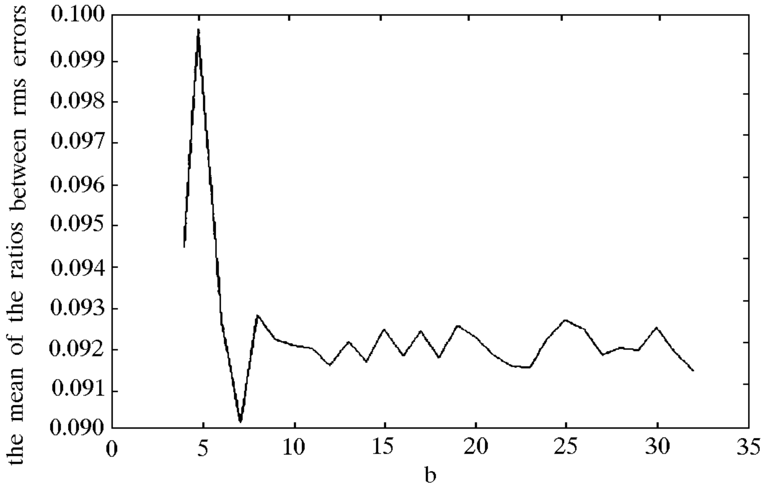

Figure 3. The mean ratio between the rms errors for indirect and direct phase.

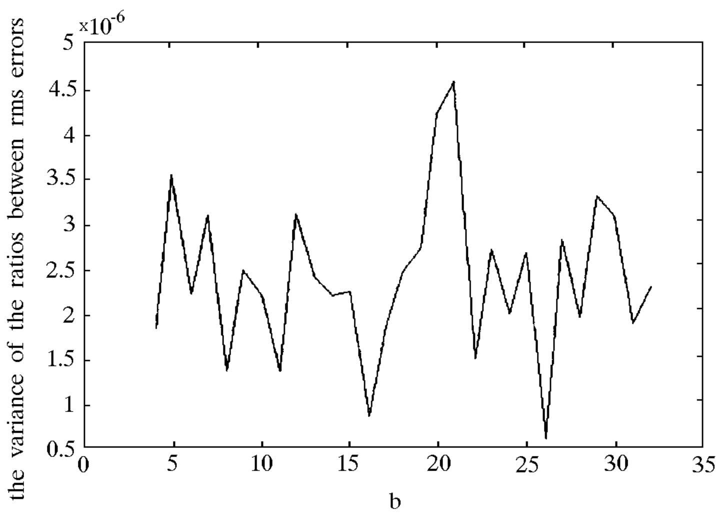

Figure 4. The variance of the ratios between the rms errors for indirect and direct phase quantization.

4. Conclusions

Examing those results, we can conclude that the ratio between the rms error for indirect quantization (12) and the rms error for direct quantization (2) is less then 0.125 $1/2^3$. This shows the fact that the quantization of the $\sin\theta$ and $\cos\theta$ function can be performed with 3 bits less then the direct quantization of phase θ with similar performances.

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