FACTA UNIVERSITATIS (NIŠ)
SERIES: ELECTRONICS AND ENERGETICS VOL. 8, NO.2 (1995), 235–253

DESIGN OF THE FIRST ORDER DIGITAL FIR DIFFERENTIATORS

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Abstract. In this paper, the original and general method for designing the first order digital FIR differentiators of even and odd length, with simultaneous approximation of the prescribed magnitude and group delay responses is presented. The proposed method represents an approach for FIR differentiator frequency response approximation, directly in the complex domain and is based on the least squares approximation method, with the originally modified eigenfilter method. It involves computing the elements and eigen-system of the quadratic, real and symmetric matrix, by the simultaneous minimization of the appropriate and originally defined quadratic measure error of the magnitude and group delay responses in the defined frequency bands. The given specifications of these two responses are incorporated in the minimization procedure. The eigenvector corresponding to the smallest eigenvalue from the computed matrix eigen-system presents the desired solution, i.e. the impulse response coefficients vector of the designed FIR differentiator. The weighting coefficients of the real and imaginary parts approximation, of the frequency response $\alpha(\omega)$ and $\beta(\omega)$ respectively, in the passband and stopband, are introduced. By the appropriate choice of these coefficient values, it is possible to affect the achieved approximation quality and accuracy. FIR differentiators, designed by this method do not posses neuther the antisymmetric feature of their impulse response coefficients, nor the strictly linear phase. Their passband group delay level is approximately constant and differs (lower or higher) from that of the corresponding linear phase FIR differentiators and can be varied in a relatively wide range. With the same length, differentiators designed by the proposed method have a lower passband magnitude response error than the corresponding mini-max differentiators. In order to illustrate its effectiveness, the numerical examples of their synthesis are also given.

Manuscript received April 26, 1995. A version of this paper was presented at the second Conference Telecommunications in Modern Satellite and Cable Services, TELSIKS'95, October 1995, Niš, Yugoslavia.

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1. Introduction

Differentiators are devices whose output signal is equal to the first or higher order derivative of the input signal. They have a broad application in various practical signal processing systems, such as: control systems, various communication systems, seismic systems, biomedical electronic devices etc. Digital FIR differentiators (DD in the subsequent text) are devices whose output signal samples are equal to the samples of the input signal derivative (of first or higher order).

In a general case, DD can be designed by using many available numerical differentiation formulas, such as : Gregory-Newton forward and backward difference formulas or Bessel, Everett and Stirling central difference formulas. Alternatively, they can be designed as nonrecursive digital filters on the basis of the frequency response of an ideal differentiator, or by Fourier series method in conjunction with the Kaiser window function.

S.Usui and I.Amidror [1] have been proposed a method for DD design, working in the region of the low frequencies, for biomedical signal processing. The main aim of this method was not the high accuracy achievement, but the possibility of the lower length DD design. Kumar and Dutta Roy have been proposed the mathematical formulas for computing the weighting coefficients of the maximally linear first order DD, working in the regions of high [2], middle [3] and low [4] frequencies of the full frequency band. The same authors, together with M.Reddy, in [5], have been performed the further expansion of this method, on the second and higher order DD design for midband frequencies. Medlin [6] and Adams [7] have been suggested a new design technique for maximally linear first and higher order DD in a general sense. S.Sunder, W.S.Lu, A.Antoniou and Y.Su [8] have been presented the synthesis method for DD satisfying prescribed specifications, by using optimization techniques. This method, somewhat, presents the hybrid between the eigenfilter method and the Chebyshev mini-max approximation method. Rabiner and Schafer [9] have been used the algorithm from reference [10] in order to design first order wide-band DD with mini-max magnitude response relative error. Equiripple nonrecursive DD design, using weighted least-squares technique is described in [11]. The results of the recent paper [12] analyze the magnitude response approximation technique with the prescribed group delay level. This technique can be generalized and reduced to the eigenfilter problem. Classic eigenfilter method [13], applied to a linear phase DD design (of first and higher order) has been presented by S.C.Pei and J.J.Shyu, [14],[15]. For the approximation of an ideal DD frequency response, they had been used a pure real functions, given in the form of weighted sums of sine (cosine) functions.

Forementioned DD design methods are techniques for designing the linear phase DD (Cases 3 and 4 of linear phase FIR filters, [16]). However, for FIR filter design, having lower time delay than the linear phase FIR filters, and approximately constant passband group delay level, it is needed to solve the complex approximation problem. L.J.Karam and J.H.McClellan [17] have been suggested the mini-max approximation technique, generalized on the real and imaginary cases of the frequency responses. This paper includes the detailed analysis of particular solutions. Using this method, it is possible to determine FIR filter transfer function with real and imaginary coefficients. Chen and Parks [18] have been used the standard linear programming algorithm. They have converted the complex Chebyshev approximation problem into a real domain by solving the overdetermined set of linear equations, using the standard linear programming techniques. Steiglitz [19] has been successfully designed this filters using linear programming technique and the weighted Chebyshev error criterion. Both of these methods are rather complicated and require a considerable computing time and computer memory space.

The method presented in this paper is more efficient and faster in that sense. In contrast to the forementioned DD design methods, this method presents an approach for DD frequency response complex function approximation directly in the complex, and not in the real domain. Also, mentioned methods clearly distinguish design cases to DD of odd and even length, in sense that for each of these two cases hold particular design formulas, i.e. forms of approximation functions. In contrast to this, the proposed method can be considered as universal in that sense, because it gives equally good results both for even and odd length DD design, without any exchange in the design procedure and formulas. Moreover, using the proposed method it is even possible, by introducing certain condition, to perform the design of odd length full-band DD. The idea for introducing and modifying the weighting coefficients of the frequency response, $\alpha(\omega)$ and $\beta(\omega)$ respectively, directly in the complex domain, is applied in the approximation error definition, and is taken from [20].

Complex approximation approach in the proposed method is performed by the simultaneous minimization of the appropriate and originally defined quadratic measure error of magnitude and group delay responses, in the defined frequency bands. Design specifications, satisfying these two responses, are incorporated in this minimization procedure, which is used for calculation the elements and eigen-system of quadratic, real and symmetric matrix. Eigenvector corresponding to the smallest eigenvalue from calculated eigen-system, presents the desired solution and design aim, i.e. the impulse response vector coefficients of the designed DD.

2. Development of the proposed method

Frequency response of an ideal DD with linear phase and length N, is given by:

$$F(\omega) = M(\omega)e^{jP(\omega)} = j\omega e^{-j\omega\tau} \quad (1a)$$

that is

$$F(\omega) = F_R(\omega) + jF_I(\omega) = \begin{cases} \omega \sin \tau\omega + j\omega \cos \tau\omega; & \omega \in \text{ passband} \\ 0; & \omega \in \text{ stopband} \end{cases} \quad (1b)$$

where in the passband:

$$F_R(\omega) = \omega \sin(\tau\omega) \quad (2a)$$

$$F_I(\omega) = \omega \cos(\tau\omega) \quad (2b)$$

and τ is the passband group delay level.

Frequency response of the designed DD, with length N and real impulse response $a(n)$; $n = 0, 1, \ldots, N-1$, is given by:

$$H(\omega) = \sum_{n=0}^{N} a(n)e^{-j\omega n} = \sum_{n=0}^{N-1} a(n)\cos(n\omega) - j\sum_{n=0}^{N-1} a(n)\sin(n\omega) = \\ = H_R(\omega) + jH_I(\omega) \quad (3)$$

where

$$H_R(\omega) = \sum_{n=0}^{N-1} a(n)\cos(n\omega) \quad (4a)$$

$$H_I(\omega) = -\sum_{n=0}^{N-1} a(n)\sin(n\omega) \quad (4b)$$

Let us define the following vectors :

$$a = [a[0],\ a[1],\ \ldots, a[N-1]]^T \quad (5)$$

$$c(\omega) = [1,\ \cos(\omega),\ \ldots,\ cos[(N-1)\omega]]^T \quad (6)$$

$$s(\omega) = [0,\ \sin(\omega),\ \ldots,\ \sin[(N-1)\omega]]^T \quad (7)$$

where the superscript T denotes the vector transpose operation.

Using relations (5), (6) and (7), expression (3) can be written in the following form:

$$H(\omega) = a^T c(\omega) - ja^T s(\omega) \tag{8}$$

while, relations (4a) and (4b), in the forms :

$$H_R(\omega) = a^T c(\omega) \tag{9a}$$

$$H_I(\omega) = -a^T s(\omega) \tag{9b}$$

respectively. Expression (8) is now used for the approximation of the desired magnitude, $M(\omega)$, and phase, $P(\omega)$, responses of ideal DD, from (1a). More exactly, in the defined frequency bands, the real, (4a), and imaginary, (4b), parts of the designed frequency response are designed to approximate the real, (2a), and imaginary, (2b), parts of the ideal DD frequency response, respectively.

This approximation is performed by the quadratic measure error minimization, in a general case defined as :

$$E = \int_R [F(\omega)H(\omega_0) - H(\omega)F(\omega_0)]^2 d\omega \tag{10}$$

where R is the desired (defined) frequency band region (passband or stopband), and ω_0 is the passband reference frequency.

Introducing the weighting coefficients, $\alpha(\omega)$ and $\beta(\omega)$, for the real and imaginary parts approximation respectively of the frequency response, in the passband and stopband, which have the following form :

$$\alpha(\omega) = \begin{cases} \alpha_p; & \omega \in \text{ passband} \\ \alpha_s; & \omega \in \text{ stopband} \end{cases} \tag{11a}$$

$$\beta(\omega) = \begin{cases} \beta_p; & \omega \in \text{ passband} \\ \beta_s; & \omega \in \text{ stopband} \end{cases} \tag{11b}$$

expression (10) can be written as :

$$\begin{aligned} E =&\alpha(\omega) \int_R [F_R(\omega)H_R(\omega_0) - H_R(\omega)F_R(\omega_0)]^2 d\omega + \\ &+\beta(\omega) \int_R [F_I(\omega)H_I(\omega_0) - H_I(\omega)F_I(\omega_0)]^2 d\omega \end{aligned} \tag{12}$$

In the above expression, in the passband case, instead of the indice R, it is needed for the integration boundaries to take the passband cutoff frequencies, and for $\alpha(\omega)$ and $\beta(\omega)$, defined values of these coefficients, α_p and β_p, respectively. Analogously, for the stopband case, instead of the indice R, for the integration boundaries is needed to take the stopband cutoff frequencies, and for $\alpha(\omega)$ and $\beta(\omega)$, their defined values α_s and β_s, respectively. It is emphasized that in the stopband hold: $F_R(\omega)=0,\ F_I(\omega)=0$.

Having in sight the previous discussion, expression (12), for $F(\omega_0)\neq 0$, can be written, in a general form:

$$E=\alpha_p E_{Rp}+\beta_p E_{Ip}+\alpha_s E_{Rs}+\beta_s E_{Is}=E_p+E_s \tag{13}$$

where

$$E_p=\alpha_p E_{Rp}+\beta_p E_I \tag{14a}$$

$$E_s=\alpha_s E_{Rs}+\beta_s E_{Is} \tag{14b}$$

and present

E_p- Total passband approximation error of the frequency response;
E_s- Total stopband approximation error of the frequency response;
E_{Rp}- Passband approximation error of the frequency response real part;
E_{Ip}- Passband approximation error of the frequency response imaginary part;
E_{Rs}- Stopband approximation error of the frequency response real part;
E_{Is}- Stopband approximation error of the frequency response imaginary part;
E- Total approximation error (of the total frequency response).

Substituting $(2a),(2b),(4a)$ and $(4b)$ in (12), expression (13) gets the following form:

$$E=a^T[\alpha_p Q_{Rp}+\beta_p Q_{Ip}+\alpha_s Q_{Rs}+\beta_s Q_{Is}]a=a^T Q_p a+a^T Q_s a \tag{15}$$

where

$$Q_p=\alpha_p Q_{Rp}+\beta_p Q_{Ip} \tag{16a}$$

$$Q_s=\alpha_s Q_{Rs}+\beta_s Q_I \tag{16b}$$

present $N\times N$ matrices of the passband (Q_p) and stopband (Q_s). By comparing equations (13) and (15), it is obvious that :

$$E_p=a^T Q_p a \tag{17b}$$

$$E_s=a^T Q_s a \tag{17b}$$

Taking into account former discussion, expression (15) can be written as follows:

$$E=a^T[Q_p+Q_s]a=a^T Q a \tag{18}$$

where

$$Q = Q_p + Q_s \tag{19}$$

is the $N \times N$ quadratic, real and symmetric matrix in general case, whose elements and eigensystem are determined. Its elements values depend on the design requirements, i.e. on given specifications.

Expression (18) for total approximation error, presents the formulation of the eigenfilter problem [14] in a least squares sense. Vector a in this expression is the eigenvector of the matrix Q. Eigenvector a of the matrix Q, corresponding to its smallest eigenvalue is the vector minimizing the error (18), and thus is the desired impulse response coefficient vector of the designed DD from equation (3).

Introduction of the weighting coefficients $\alpha(\omega)$ and $beta(\omega)$, for approximation the real and imaginary parts of the frequency response respectively, enables to affect the achieved approximation quality and accuracy, by appropriate choice of their values in the corresponding frequency bands. Calculating integrals in equation (12), expressions for the elements of matrices Q_p and Q_s are obtained in a compact, explicit, agebraic form:

$$q_p(n,m) = \beta_p \Bigg\{ \Bigg(\frac{\omega^3 \cos(m\omega_0 - n\omega_0)}{12} + \frac{\omega \cos(2\tau\omega) \cos(m\omega_0 - n\omega_0)}{8\tau^2} -$$

$$\frac{\omega^3 \cos(m\omega_0 + n\omega_0)}{12} - \frac{\omega \cos(2\tau\omega) \cos(m\omega_0 + n\omega_0)}{8\tau^2} +$$

$$\frac{{\omega_0}^2 \sin((m-n)\,\omega)}{4\,(m-n)} + \frac{{\omega_0}^2 \cos(2\tau\omega_0) \sin((m-n)\,\omega)}{4\,(m-n)} -$$

$$\frac{{\omega_0}^2 \sin((m+n)\,\omega)}{4\,(m+n)} - \frac{{\omega_0}^2 \cos(2\tau\omega_0) \sin((m+n)\,\omega)}{4\,(m+n)} -$$

$$\frac{\cos(m\omega_0 - n\omega_0) \sin(2\tau\omega)}{16\tau^3} + \frac{\omega^2 \cos(m\omega_0 - n\omega_0) \sin(2\tau\omega)}{8\tau} +$$

$$\frac{\cos(m\omega_0 + n\omega_0) \sin(2\tau\omega)}{16\tau^3} - \frac{\omega^2 \cos(m\omega_0 + n\omega_0) \sin(2\tau\omega)}{8\tau} +$$

$$\frac{\omega\omega_0 \cos((n-\tau)\,\omega) \sin(m\omega_0 - \tau\omega_0)}{4\,(n-\tau)} + \frac{\omega\omega_0 \cos((n+\tau)\,\omega) \sin(m\omega_0 - \tau\omega_0)}{4\,(n+\tau)} -$$

$$\frac{\omega_0 \sin((n-\tau)\,\omega) \sin(m\omega_0 - \tau\omega_0)}{4(n-\tau)^2} - \frac{\omega_0 \sin((n+\tau)\,\omega) \sin(m\omega_0 - \tau\omega_0)}{4(n+\tau)^2} +$$

$$\frac{\omega\omega_0 \cos((m+\tau)\,\omega) \sin(n\omega_0 - \tau\omega_0)}{4\,(m+\tau)} - \frac{\omega_0 \sin((m-\tau)\,\omega) \sin(n\omega_0 - \tau\omega_0)}{4(m-\tau)^2} -$$

$$\frac{\omega_0 \sin((m+\tau)\,\omega)\sin(n\omega_0-\tau\omega_0)}{4(m+\tau)^2}+\frac{\omega\omega_0\cos((n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4\,(n+\tau)}-$$
$$\frac{\omega_0 \sin((n-\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4(n-\tau)^2}-\frac{\omega_0\sin((n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4(n+\tau)^2}+$$
$$\frac{\omega\omega_0 \cos((m-\tau)\,\omega)\sin(n\omega_0-\tau\omega_0)}{4\,(m-\tau)}+\frac{\omega\omega_0\cos((n-\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4\,(n-\tau)}+$$
$$\frac{\omega\omega_0 \cos((m-\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4\,(m-\tau)}+\frac{\omega\omega_0\cos((m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4\,(m+\tau)}-$$
$$\frac{\omega_0 \sin((m-\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4(m-\tau)^2}-$$
$$\left.\left.\frac{\omega_0 \sin((m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4(m+\tau)^2}\right)\right|_{\omega_{lp}}^{\omega_{hp}}\Bigg\}+$$
$$\alpha_p\Bigg\{\Bigg(\frac{\omega^3\cos(m\omega_0-n\omega_0)}{12}-\frac{\omega\cos(2\tau\omega)\cos(m\omega_0-n\omega_0)}{8\tau^2}+$$
$$\frac{\omega^3\cos(m\omega_0+n\omega_0)}{12}-\frac{\omega\cos(2\tau\omega)\cos(m\omega_0+n\omega_0)}{8\tau^2}+$$
$$\frac{{\omega_0}^2\sin((m-n)\,\omega)}{4\,(m-n)}-\frac{{\omega_0}^2\cos(2\tau\omega_0)\sin((m-n)\,\omega)}{4\,(m-n)}+$$
$$\frac{{\omega_0}^2\sin((m+n)\,\omega)}{4\,(m+n)}-\frac{{\omega_0}^2\cos(2\tau\omega_0)\sin((m+n)\,\omega)}{4\,(m+n)}+$$
$$\frac{\cos(m\omega_0-n\omega_0)\sin(2\tau\omega)}{16\tau^3}-\frac{\omega^2\cos(m\omega_0-n\omega_0)\sin(2\tau\omega)}{8\tau}+$$
$$\frac{\cos(m\omega_0+n\omega_0)\sin(2\tau\omega)}{16\tau^3}-\frac{\omega^2\cos(m\omega_0+n\omega_0)\sin(2\tau\omega)}{8\tau}+$$
$$\frac{\omega\omega_0\cos((n+\tau)\,\omega)\sin(-(m\omega_0)+\tau\omega_0)}{4\,(n+\tau)}-$$
$$\frac{\omega_0\sin((-n+\tau)\,\omega)\sin(-(m\omega_0)+\tau\omega_0)}{4(-n+\tau)^2}-$$

$$\frac{\omega_0 \sin((n+\tau)\,\omega)\sin(-(m\omega_0)+\tau\omega_0)}{4(n+\tau)^2}+$$
$$\frac{\omega\omega_0 \cos((n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4\,(n+\tau)}-$$
$$\frac{\omega_0 \sin((-n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4(-n+\tau)^2}-$$
$$\frac{\omega_0 \sin((n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4(n+\tau)^2}+$$
$$\frac{\omega\omega_0 \cos((m+\tau)\,\omega)\sin(-(n\omega_0)+\tau\omega_0)}{4\,(m+\tau)}-$$
$$\frac{\omega_0 \sin((-m+\tau)\,\omega)\sin(-(n\omega_0)+\tau\omega_0)}{4(-m+\tau)^2}-$$
$$\frac{\omega_0 \sin((m+\tau)\,\omega)\sin(-(n\omega_0)+\tau\omega_0)}{4(m+\tau)^2}+$$
$$\frac{\omega\omega_0 \cos((-m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4\,(-m+\tau)}+$$
$$\frac{\omega\omega_0 \cos((m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4\,(m+\tau)}-$$
$$\frac{\omega_0 \sin((-m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4(-m+\tau)^2}-$$
$$\frac{\omega\omega_0 \cos((-n+\tau)\,\omega)\sin(-(m\omega_0)+\tau\omega_0)}{4\,(-n+\tau)}+$$
$$\frac{\omega\omega_0 \cos((-n+\tau)\,\omega)\sin(m\omega_0+\tau\omega_0)}{4\,(-n+\tau)}+$$
$$\frac{\omega\omega_0 \cos((-m+\tau)\,\omega)\sin(-(n\omega_0)+\tau\omega_0)}{4\,(-m+\tau)}+$$
$$\left.\left.\frac{\omega_0 \sin((m+\tau)\,\omega)\sin(n\omega_0+\tau\omega_0)}{4(m+\tau)^2}\right)\right|_{\omega_{lp}}^{\omega_{hp}}\Bigg\},$$
$$0 \le n, m \le N-1, n \ne m \qquad (20a)$$

that is, for the matrix Q_p elements outside its main diagonal.

$$
\begin{aligned}
q_p(k,k) = \alpha_p \Bigg\{ \Bigg(& \frac{\omega^3}{12} + \frac{\omega {\omega_0}^2}{4} - \frac{\omega \cos(2\tau\omega)}{8\tau^2} + \\
& \frac{\omega^3 \cos(2k\omega_0)}{12} - \frac{\omega \cos(2\tau\omega)\cos(2k\omega_0)}{8\tau^2} - \frac{\omega {\omega_0}^2 \cos(2\tau\omega_0)}{4} + \\
& \frac{{\omega_0}^2 \sin(2k\omega)}{8k} + \frac{\sin(2\tau\omega)}{16\tau^3} - \frac{\omega^2 \sin(2\tau\omega)}{8\tau} + \\
& \frac{\cos(2k\omega_0)\sin(2\tau\omega)}{16\tau^3} - \frac{\omega^2 \cos(2k\omega_0)\sin(2\tau\omega)}{8\tau} - \\
& \frac{{\omega_0}^2 \sin(2k\omega - 2\tau\omega_0)}{16k} + \frac{\omega\omega_0 \cos((-k+\tau)\,\omega)\sin(-(k\omega_0)+\tau\omega_0)}{2\,(-k+\tau)} + \\
& \frac{\omega\omega_0 \cos((k+\tau)\,\omega)\sin(-(k\omega_0)+\tau\omega_0)}{2\,(k+\tau)} - \\
& \frac{\omega_0 \sin((-k+\tau)\,\omega)\sin(-(k\omega_0)+\tau\omega_0)}{2{(-k+\tau)}^2} - \\
& \frac{\omega_0 \sin((k+\tau)\,\omega)\sin(-(k\omega_0)+\tau\omega_0)}{2{(k+\tau)}^2} + \\
& \frac{\omega\omega_0 \cos((-k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(-k+\tau)} + \\
& \frac{\omega\omega_0 \cos((k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(k+\tau)} - \\
& \frac{\omega_0 \sin((-k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2{(-k+\tau)}^2} - \\
& \frac{\omega_0 \sin((k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2{(k+\tau)}^2} - \frac{{\omega_0}^2 \sin(2k\omega+2\tau\omega_0)}{16k} \Bigg) \Bigg\} \Bigg|_{\omega_{lp}}^{\omega_{hp}} +
\end{aligned}
$$

$$\beta_p\Bigg\{\Bigg(\frac{\omega^3}{12}+\frac{\omega\omega_0^2}{4}+\frac{\omega\cos(2\tau\omega)}{8\tau^2}-\frac{\omega^3\cos(2k\omega_0)}{12}-$$
$$\frac{\omega\cos(2\tau\omega)\cos(2k\omega_0)}{8\tau^2}+\frac{\omega\omega_0^2\cos(2\tau\omega_0)}{4}-\frac{\omega_0^2\sin(2k\omega)}{8k}-$$
$$\frac{\sin(2\tau\omega)}{16\tau^3}+\frac{\omega^2\sin(2\tau\omega)}{8\tau}+\frac{\cos(2k\omega_0)\sin(2\tau\omega)}{16\tau^3}-\frac{\omega^2\cos(2k\omega_0)\sin(2\tau\omega)}{8\tau}-$$
$$\frac{\omega_0^2\sin(2k\omega-2\tau\omega_0)}{16k}+\frac{\omega\omega_0\cos((k-\tau)\,\omega)\sin(k\omega_0-\tau\omega_0)}{2\,(k-\tau)}+$$
$$\frac{\omega\omega_0\cos((k+\tau)\,\omega)\sin(k\omega_0-\tau\omega_0)}{2\,(k+\tau)}-\frac{\omega_0\sin((k-\tau)\,\omega)\sin(k\omega_0-\tau\omega_0)}{2\,(k-\tau)^2}-$$
$$\frac{\omega_0\sin((k+\tau)\,\omega)\sin(k\omega_0-\tau\omega_0)}{2\,(k+\tau)^2}+\frac{\omega\omega_0\cos((k-\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(k-\tau)}+$$
$$\frac{\omega\omega_0\cos((k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(k+\tau)}-\frac{\omega_0\sin((k-\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(k-\tau)^2}-$$
$$\frac{\omega_0\sin((k+\tau)\,\omega)\sin(k\omega_0+\tau\omega_0)}{2\,(k+\tau)^2}-\frac{\omega_0^2\sin(2k\omega+2\tau\omega_0)}{16k}\Bigg)\Bigg|_{\omega_{lp}}^{\omega_{hp}}\Bigg\}$$
$$k=1,2,\ldots,N-1 \tag{20b}$$

for $k=1,2,\ldots,N-1$, i.e. for the matrix Q_p elements on its main diagonal.

$$q_p(0,0)=\alpha_p\Bigg\{\Bigg(\frac{\omega^3}{6}+\frac{\omega{\omega_0}^2}{2}-\frac{\omega\cos(2\tau\omega)}{4\tau^2}-\frac{\omega{\omega_0}^2\cos(2\tau\omega_0)}{2}-$$
$$\frac{\omega_0\cos(\tau\omega-\tau\omega_0)}{\tau^2}+\frac{\omega_0\cos(\tau\omega+\tau\omega_0)}{\tau^2}+\frac{\sin(2\tau\omega)}{8\tau^3}- \tag{20c}$$
$$\frac{\omega^2\sin(2\tau\omega)}{4\tau}+\frac{\omega\omega_0\sin(-(\tau\omega)+\tau\omega_0)}{\tau}+\frac{\omega\omega_0\sin(\tau\omega+\tau\omega_0)}{\tau}\Bigg)\Bigg\}\Bigg|_{\omega lp}^{\omega_{hp}}$$

for the first element of matrix Q_p.

$$
\begin{aligned}
q_s(n,m) = {} & \alpha_{s1}\left(\frac{\sin(n-m)\omega}{2(n-m)}+\frac{\sin(n+m)\omega}{2(n+m)}\right)(\omega_0\sin\tau\omega_0)^2\Bigg|_{\omega_{l1s}}^{\omega_{h1s}} + \\
& \alpha_{s2}\left(\frac{\sin(n-m)\omega}{2(n-m)}+\frac{\sin(n+m)\omega}{2(n+m)}\right)(\omega_0\sin\tau\omega_0)^2\Bigg|_{\omega_{l2s}}^{\omega_{h2s}} + \\
& \beta_{s1}\left(\frac{\sin(n-m)\omega}{2(n-m)}-\frac{\sin(n+m)\omega}{2(n+m)}\right)(\omega_0\cos\tau\omega_0)^2\Bigg|_{\omega_{l1s}}^{\omega_{h1s}} + \\
& \beta_{s2}\left(\frac{\sin(n-m)\omega}{2(n-m)}-\frac{\sin(n+m)\omega}{2(n+m)}\right)(\omega_0\cos\tau\omega_0)^2\Bigg|_{\omega_{l2s}}^{\omega_{h2s}} +
\end{aligned}
\qquad (21a)
$$

for $0 \le n,\ m \le N-1;\ n \ne m$, i.e. for the matrix Q_s elements outside its main diagonal.

$$
\begin{aligned}
q_s(k,k) = {} & \alpha_{s1}\left(\left(\frac{1}{2}-\frac{\sin 2k\omega}{4k}\right)(\omega_0\sin\tau\omega_0)^2\right)\Bigg|_{\omega_{ls1}}^{\omega_{hs1}} + \\
& \alpha_{s2}\left(\left(\frac{1}{2}-\frac{\sin 2k\omega}{4k}\right)(\omega_0\sin\tau\omega_0)^2\right)\Bigg|_{\omega_{ls2}}^{\omega_{hs2}} + \\
& \beta_{s1}\left(\left(\frac{1}{2}+\frac{\sin 2kx}{4k}\right)(\omega_0\cos\tau\omega_0)^2\right)\Bigg|_{\omega_{ls1}}^{\omega_{hs1}} + \\
& \beta_{s2}\left(\left(\frac{1}{2}+\frac{\sin 2kx}{4k}\right)(\omega_0\cos\tau\omega_0)^2\right)\Bigg|_{\omega_{ls2}}^{\omega_{hs2}}
\end{aligned}
\qquad (21b)
$$

for $k = 1, 2, \dots, N-1$, i.e. for the matrix Q_s elements on its main diagonal.

$$
q_s(0,0) = \alpha_{s2}(\pi-\omega_{s2})(\omega_0\sin\tau\omega_0)^2 + \alpha_{s1}\omega_{s1}(\omega_0\sin\tau\omega_0)^2 \qquad (21c)
$$

for the first element of matrix Q_s.

Expressions (20)–(21) are derived for the case of band–pass digital differentiator, but they are general enough and include all other particular solutions: for low–pass digital differentiator is

$$
\alpha_{s1}=0,\ \alpha_{s2}=\alpha_s;\quad \beta_{s1}=0,\ \beta_{s2}=\beta_s;\quad \omega_{lp}=0,\ \omega_{hs}=\pi
$$

for high pass digital differentiator is:

$$
\alpha_{s1}=\alpha_s,\ \alpha_{s2}=0;\quad \beta_{s1}=\beta_s,\ \beta_{s2}=0;\quad \omega_{hp}=\pi,\ \omega_{lp}=0
$$

and for full–band digital differentiator is

$$\alpha_{s1} = \alpha_{s2} = 0; \quad \beta_{s1} = \beta_{s2} = 0; \quad \omega_{lp} = 0, \ \omega_{hp} = \pi$$

which means that in the full–band case is

$$q_s(n,m) \equiv 0, \quad \forall \, n, m$$

i.e. the matrix Q_s is the zero–matrix.

Indices ω_h and ω_l in above expressions respectively denote the upper and lower cutoff frequencies of the passband (subscript p) and stopband (subscript s).

Basing on the equation (19), the matrix Q elements are, finally, given by:

$$q(n,m) = q_p(n+m) + q_s(n,m) \tag{22a}$$

$$q(k,k) = q_p(k,k) + q_s(k,k) \tag{22b}$$

$$q(1,1) = q_p(1,1) + q_s(1,1) \tag{22c}$$

for the elements outside (22a), and on (22b) its main diagonal, so as its first element (22c). It can be seen, from equations (20)–(22), that the matrix Q elements values depend on the given parameter specifications, which are:

N - designed DD length ;
ω_{hp} - upper passband cutoff frequency;
ω_{lp} - lower passband cutoff frequency;
ω_{hs} - upper stopband cutoff frequency;
ω_{ls} - lower stopband cutoff frequency;
α_p - real part approximation weighting coefficient in the passband;
α_s - real part approximation weighting coefficient in the stopband;
β_p - imaginary part approximation weighting coefficient in the passband;
β_s - imaginary part approximation weighting coefficient in the stopband;
τ - desired passband group delay level;
ω_0 - passband reference frequency.

By introducing the condition that the specification parameter τ has the value which is not an integer, it is even possible to design the odd length full-band DD. Using presented method, design of the large number of various first order DD types with even and odd length is performed, such as: full–band, low–pass, band–pass and high–pass. Their design examples with corresponding specification parameters values are presented in the next section.

3. Design examples

Example 1

Design of FIR full–band differentiator with following specification parameters values: $N = 32$; $\tau = 9.5$; $\omega_0 = 0.5\pi$; $\alpha = 0.01$; $\beta = 0.99$; $\omega_{lp} = 0$; $\omega_{hp} = \pi$.

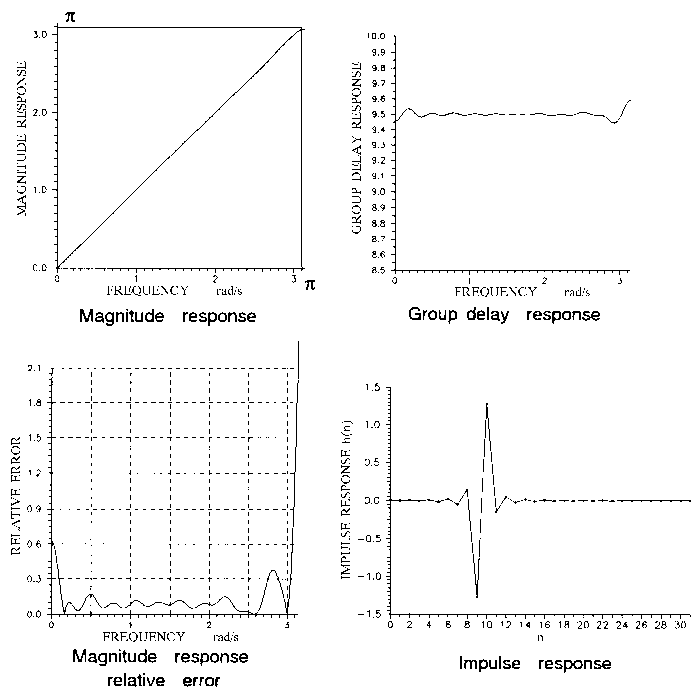

Fig. 1. FIR full–band differentiator

Example 2

Design of FIR low–pass differentiator with following specification parameters values: $N = 31$; $\tau = 11.5$; $\omega_0 = 0.15\pi$; $\alpha_p = 0.45$; $\beta_p = 0.55$; $\alpha_s = 0.1\alpha_p$; $\beta_s = 0.1\beta_p$; $\omega_{lp} = 0$; $\omega_{hp} = 0.3\pi$; $\omega_{ls} = 0.4\pi$; $\omega_{hs} = \pi$.

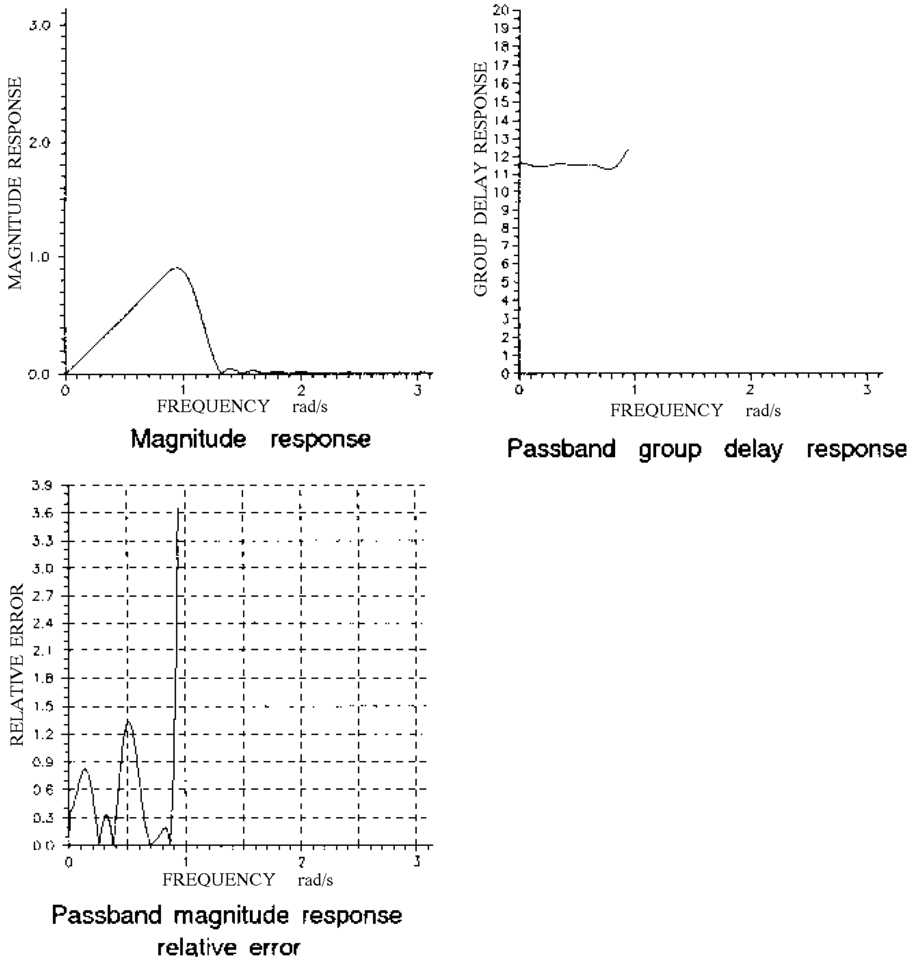

Fig. 2. FIR low–pass diferentiator

Example 3

Design of FIR high–pass differentiator with following specification parameters values: $N = 32$; $\tau = 9.5$; $\omega_0 = 0.85\pi$; $\alpha_p = 0.55$; $\beta_p = 0.45$; $\alpha_s = 0.1\alpha_p$; $\beta_s = 0.1\beta_p$; $\omega_{lp} = 0.7\pi$; $\omega_{hp} = \pi$; $\omega_{ls} = 0$; $\omega_{hs} = 0.6\pi$.

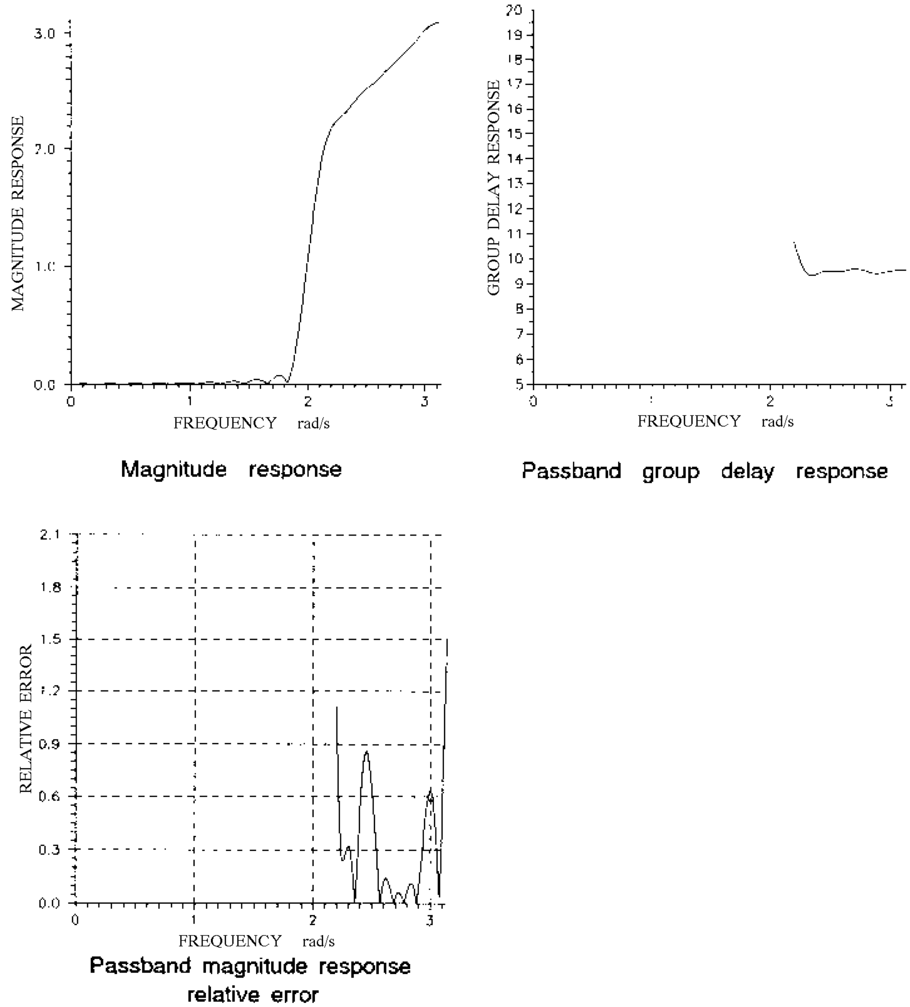

Fig. 3. FIR high–pass diferentiator

Example 4

Design of FIR band–pass differentiator with following specification parameters values: $N = 31$; $\tau = 15.5$; $\omega_0 = 0.5\pi$; $\alpha_p = \beta_p = 0.5$; $\alpha_{s1} = 10\alpha_p$; $\beta_{s1} = 10\beta_p$; $\alpha_{s2} = 0.1\alpha_p$; $\beta_{s2} = 0.1\beta_p$; $\omega_{ls1} = 0$; $\omega_{hs1} = 0.1\pi$; $\omega_{lp} = 0.2\pi$; $\omega_{hp} = 0.8\pi$; $\omega_{ls2} = 0.9\pi$; $\omega_{hs2} = \pi$.

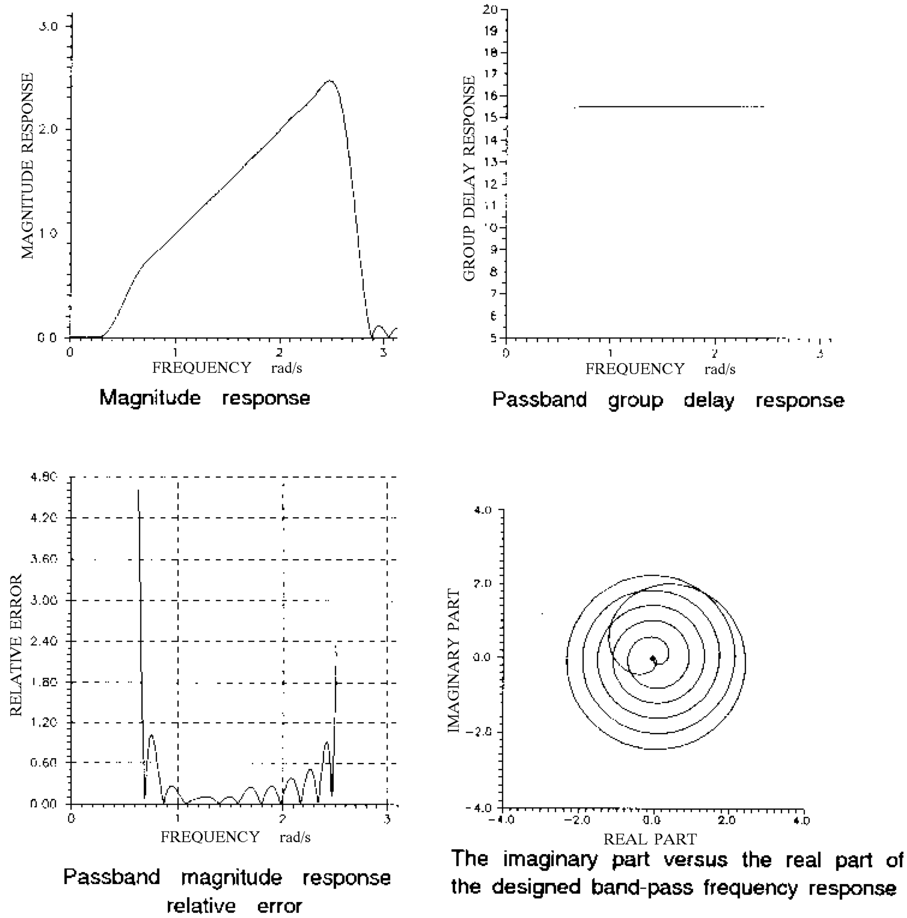

Fig. 4. FIR band–pass diferentiator

4. Conclusion

The design method presented in this paper contains the new contribution to the global design of the first order FIR differentiators with prescribed magnitude and group delay responses. In comparison to analogous methods, this method is very fast, easy and effective, because it does not involve iterative calculations. This method presents the approach for the FIR differentiator frequency response approximation, directly in the complex, and not in the real domain, and is based on the eigenfilter method modified in the original manner. The elements of matrix Q are determined in the compact, explicit, algebraic form. FIR differentiators, designed by this method posses neither the antisymmetric feature of their impulse response coefficients, nor the strictly linear phase. They have approximately constant passband group delay level, which differs (lower or higher) from that of the corresponding linear-phase FIR differentiators and can be varied in a relatively wide range. Their passband magnitude response error, with the same length N, is lower than that of the corresponding mini-max FIR differentiators. However, the problem of the optimal solution detection is open and depends on design specifications, because the performed research has shown that, in principle, better group delay response corresponds to a higher passband magnitude response error, and vice versa.

REFERENCES

1. S. Usui, I. Amidror: *Digital low-pass differentiation for biological signal processing.* IEEE Trans. on Biomedical Eng.,Vol.BME-29, No.10 , pp.686-693, 1982 .
2. B. Kumar, S.C. Dutta Roy: *Maximally linear FIR digital differentiators for high frequencies.* IEEE Trans. on Circuit and Systems ,Vol.36, No.6, pp.890-895, 1989.
3. B. Kumar, S.C. Dutta Roy: *Design of efficient FIR digital differentiators and Hilbert Transformers for midband Frequency ranges.* Int.J.Circuit Theory Appl., Vol.17,(4), pp.483-488, 1989.
4. B. Kumar, S.C. Dutta Roy: *Design of digital differentiators for low frequencies.* Proc .IEEE, Vol. 76,(3), pp.287-289, 1988.
5. M.R. Reddy, S.C. Dutta-Roy, B. Kumar: *Design of efficient second and higher degree FIR digital differentiators for midband frequencies.* IEEE Proceedings-G, Vol.138, No.1, pp.29-33, 1991.
6. G.W. Medlin: *A design technique for high order digital differentiators.* Proceedings of the ICASSP, Albuqerque,USA, 1990.
7. G.W. Medlin, J.W. Adams: *A design technique for maximally linear differentiators.* Proceedings of the IEEE International Conference on Acoustic, Speech and Signal Processing, Vol.2., pp.825-828, Glasgow, United Kingdom, 1989.
8. S. Sunder, W.S. Lu, A. Antoniu, Y. Su: *Design of digital differentiators satisfying prescribed specifications using optimization techniques.* IEE Proceedings - G. Vol. 138 , No.3 , pp.315-320, 1991.
9. L.R. Rabiner, R.W. Schafer: *On the behaviour of minimax relative error FIR digital differentiators.*Bell Syst.Tech.J., 1974, 53, pp.333-362.
10. J.H. McClellan, T.W. Parks, L.R. Rabiner: *A computer program for designing optimum FIR linear phase digital filters.* IEEE Trans.Audio Electroacoust., Vol.AU-21,pp.506-526, 1973.
11. S. Sunder, V. Ramachandran: *Design of equiripple nonrecursive digital differentiators and Hilbert transformers using weighted least-squares technique.* IEE Trans. on Signal Processing , Vol.42 No.9, pp.2504-2509, 1994.

12. M.H. Er, C.K. Siew: *Design of FIR filters using quadratic Programming approach.* IEEE Trans. on Circuits and Systems-II : Analog and Digital Signal Processing, Vol.42, No.3, pp. 217-220, March 1995.

13. P.P. Vaidyanathanan, T.Q. Nguyen: *Eigenfilters : A new approach to least squares FIR filter design and application.* IEEE Trans. on Circuit and Systems, Vol.CAS-34, pp.11-23, 1987.

14. S.C. Pei, J.J. Shyu: *Design of FIR Hilbert transformers and differentiators by eigenfilter.* IEEE Trans. on Circuits and Systems, Vol.CAS-35, pp.1457-1461, 1989.

15. S.C. Pei, J.J. Shyu: *Eigenfilter design of higher order digital differentiators.* IEEE Trans.on Acoust., Speech and Signal Processing, Vol.ASSP-37, pp.505-511, 1989.

16. L.R. Rabiner, B. Gold: *Theory and application of digital signal processing.* Englewood Cliffs, NJ. : Prentice-Hall , 1975.

17. L.J. Karam, J.H. McClellan: *Complex Chebyshev approximation for FIR filter design.* IEEE Trans. on Circuits and Systems-II : Analog and Digital Signal Processing, Vol.42, No.3, pp. 207-216, March 1995.

18. X. Chen, T.W. Parks: *Design of FIR filters in the complex domain.* IEEE Transactions on Acoust.,Speech and Signal Processing, Vol.ASSP-35, No.2 , pp.144-153, 1987.

19. K. Steiglitz: *Design of FIR digital phase networks.* IEEE Transactions on Acoustics, Speech and Signal Process.,Vol.ASSP-29,No.2, pp.171-176, 1981.

20. V.D. Pavlović: *Direct synthesis of filter transfer functions.* IEE Proceedings Vol.131, Pt.G. No.4, pp.156-160, August 1984.