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A METHOD FOR SPACE-FREQUENCY IMAGE ANALYSIS

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Abstract. An efficient method for two-dimensional time-frequency analysis, derived from the analysis of the Wigner Distribution (WD) defined in the frequency domain, is presented. This method provides some substantial advantages over the WD. The well known cross-terms effects are reduced or completely removed; the oversampling of signals is shown to be unnecessary; the computation time can be significantly reduced, as well. The theory is illustrated by a numerical example.

1. Introduction

Time-frequency signal analysis attracted the attention of many researchers in the recent years, which is documented in [1],[2], and references therein.

A two-dimensional time-frequency distribution (TFD) of a signal $f(x,y)$ should satisfie the following basic properties:

$$\frac{1}{(2\pi)^2}\int\int\int\int MTFD(\omega_x,\omega_y,x,y)d\omega_x d\omega_y dxdy = E_f \tag{1}$$

$$\begin{aligned}\frac{1}{(2\pi)^2}\int\int MTFD(\omega_x,\omega_y,x,y)d\omega_x d\omega_y &= \mid f(x,y)\mid^2\\ \times\int\int MTFD(\omega_x,\omega_y,x,y)dxdy &= \mid F(\omega_x,\omega_y)\mid^2\end{aligned} \tag{2}$$

where: E_f denotes the energy of $f(x,y)$; $F(\omega_x,\omega_y)$ is a two-dimensional Fourier transform of $f(x,y)$; $d\omega_x d\omega_y$ and $dxdy$ are two-dimensional differential elements. An infinite number of distributions satisfying (1) and (2)

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can be defined - the two-dimensional extension of the Cohen class of distributions, [1].

The above relations do not provide any information about the local distribution of energy at a point $(\omega_x, \omega_y, x, y)$; therefore, it is necessary to define some more specific requirements as compared to (1) and (2), and they are presented in Section 2. Based on these requirements, an efficient two-dimensional time-frequency distribution is developed, extending our previously defined one-dimensional method, [4,6,7,8].

2. Local frequency presentation

Let us take a two-dimensional signal:

$$f(x,y) = g(x,y)e^{j\Phi(x,y)} \tag{3}$$

with $g(x,y)$ slow-varying two-dimensional function. The associated local frequency at a point is defined as $(\omega_x, \omega_y) = \nabla\Phi(x,y) = \vec{i}_x\Phi'_x + \vec{i}_y\Phi'_y$, where $\Phi'_x \equiv \partial\Phi(x,y)/\partial x$ and $\Phi'_y \equiv \partial\Phi(x,y)/\partial y$, with ∇ denoting the Hamiltonian operator. The ideal TFD for the above signal has the local power $\mid g(x,y) \mid^2$ concentrated at the local frequency:

$$ITFD(\omega_x, \omega_y, x, y) = (2\pi)^2 \mid g(x,y) \mid^2 \delta(\omega_x - \Phi'_x, \omega_y - \Phi'_y). \tag{4}$$

This form has already been defined and used in the one-dimensional case, [5,7,11]. We will now compare the commonly used TFD with the one defined by (4).

2.1 Two-dimensional short time Fourier transform

The Short Time Fourier Transform (STFT) of the signal $f(x,y)$ is defined by

$$STFT(\omega_x, \omega_y, x, y) = \int\int f(x+\alpha, y+\beta)w^*(\alpha,\beta)e^{-(j\omega_x\alpha+\omega_y\beta)}d\alpha d\beta \tag{5}$$

where $w^*(\alpha,\beta)$ denotes a two-dimensional, usually even and real-valued, window function. It will be assumed that $w(\alpha,\beta) = 0$ holds outside the bounded two-dimensional region $D \subset R^2$.

Substituting signal (3) into (5) and expanding $\Phi(x+\alpha, y+\beta)$ into a

Taylor series[1] around (x, y), we obtain:

$$STFT(\omega_x, \omega_y, x, y) = \frac{1}{(2\pi)^2} g(x, y) e^{j\Phi(x,y)} \delta(\omega_x - \Phi'_x, \omega_y - \Phi'_y)$$
$$**_{\omega_x \omega_y} W(\omega_x, \omega_y) **_{\omega_x, \omega_y} FT \left[e^{j \frac{(\vec{v}\nabla)^2}{2!} \Phi(x_1, y_1)} \right] \tag{6}$$

where $**_{\omega_x, \omega_y}$ denotes a two-dimensional convolution operator with respect to ω_x, ω_y, and $g(x, y)$ is treated as a constant inside the window $w(\alpha, \beta)$, i.e., $g(x + \alpha, y + \beta) w(\alpha, \beta) \cong g(x, y) w(\alpha, \beta)$.

If the second and higher-order partial derivatives of $\Phi(x, y)$ may be neglected in (6), then the associated spectrogram (squared magnitude of the STFT) becomes:

$$SPEC(\omega_x, \omega_y, x, y) = | STFT(\omega_x, \omega_y, x, y) |^2 = | g(x, y) |^2 W^2(\omega_x - \Phi'_x, \omega_y - \Phi'_y). \tag{7}$$

Observe that the spectrogram (7) exhibits all desirable properties of the ideal distribution (4), provided the behavior of $W^2(\omega_x, \omega_y)$ is close to $(2\pi)^n \delta(\omega_x, \omega_y)$. If, on the other hand, higher-order partial derivatives are not negligible, the spectrogram contains artifacts even for the ideal behavior of $W(\omega_x, \omega_y)$.

2.2 Two-dimensional Wigner distribution

Pseudo Wigner Distribution (PWD):

$$PWD(\omega_x, \omega_y, x, y) = \int\int f(x + \frac{\alpha}{2}, y + \frac{\beta}{2})$$
$$\times f^*(x - \alpha/2, y - \beta/2) w_s(\alpha, \beta) e^{-(j\omega_x \alpha + \omega_y \beta)} d\alpha d\beta \tag{8}$$

with $(\alpha, \beta) = w(\alpha/2, \beta/2) w^*(-\alpha/2, \beta/2)$ is very commonly used in the time-frequency analysis. For signals (3), upon substitution in (8) and expansion

[1] Taylor series for an n-dimensional function is of the form:

$$\Phi(\vec{r} + \vec{v}) = \sum_{i=0}^{m-1} \frac{(\vec{v}\nabla)^i}{i!} \Phi(\vec{r}) + \frac{(\vec{v}\nabla)^m}{m!} \Phi(\vec{r}_1)$$

with $\vec{r}_1 = \vec{r} + \vec{v}_1$, and $0 < v_{1h} < v_h$ for each $h = 1, 2, \ldots, n$.

of $\Phi(x+\alpha/2, y+\beta/2)$ and $\Phi(x-\alpha/2, y-\beta/2)$ into Taylor series, the following expression for the PWD is obtained:

$$PWD(\omega_x,\omega_y,x,y) = \frac{1}{(2\pi)^2} \mid g(x,y) \mid^2 \delta(\omega_x - \Phi'_x, \omega_y - \Phi'_y)$$
$$** _{\omega_x\omega_y} W_s(\omega_x,\omega_y) ** _{\omega_x\omega_y} FT \left[e^{j2\frac{(\vec{v}\nabla)^2}{3!L^2}\Phi(x_1,y_1)} \right]. \quad (9)$$

The PWD provides an ideal time-frequency representation if the third and higher-order partial derivatives of $\Phi(x,y)$ are negligible. An improvement over the STFT is obvious.

3. Two-dimnensional L-Wigner distribution

Using the L-Wigner distribution (LWD), defined by:

$$LWD(\omega_x,\omega_y,x,y) = \int\int f^L(x+\alpha/2L, y+\beta/2L)$$
$$\times f^{*L}(x-\alpha/2L, y-\beta/2L) w_L(\alpha,\beta) e^{-j\omega_x\alpha+\omega_y\beta} d\alpha d\beta$$

with $w_L(\alpha,\beta) = w(\alpha/2L, \beta/2L) w^*(-\alpha/2L, \beta/2L)$, further improvement of energy concentration may be achieved. For the signal of the form (3) we have:

$$LWD(\omega_x,\omega_y,x,y) = \frac{1}{(2\pi)^2} \mid g(x,y) \mid^{2L} \delta(\omega_x - \Phi'_x, \omega_y - \Phi'_y)$$
$$** _{\omega_x\omega_y} W_s(\omega_x,\omega_y) ** _{\omega_x\omega_y} FT \left[e^{j\frac{(\vec{v}\nabla)^2}{2L}\Phi(x_1,y_1)} \right]$$

Taking $L = 1$, the WD is obtained. The properties of LWD can be easily derived following the one-dimensional case given in [5,18,19].

4. Analysis of multicomponent signals

We will now consider a two-dimensional multicomponent signal given by

$$f(x,y) = \sum_{i=1}^{p} g_i(x,y) e^{j\Phi_i(x,y)} \quad (10)$$

where the functions $g_i(x,y), i = 1,..,p$, belong to the same class as $g(x,y)$ in (3).

As before, the spectrogram for signal (10) is:

$$SPEC(x,y,\omega_x,\omega_y) = \sum_{i=1}^{p}\sum_{k=1}^{p} g_i(x,y)g_k(x,y)e^{j[\Phi_i(x,y)-\Phi_k(x,y)]} \\ \times W[\omega_x - \Phi'_{xi}, \omega_y - \Phi'_{yi}]W^*[\omega_x - \Phi'_{xk}, \omega_y - \Phi'_{yk}] \tag{11}$$

where the artifacts due to higher-order partial derivatives of $\Phi_i(x,y), i = 1,2,\ldots,p$, are neglected, i.e. $\nabla\Phi_i(x,y)$ is treated as a constant vector inside $w(x,y)$.

The cross-terms are absent from the spectrogram, provided the condition:

$$W[\omega_x - \Phi'_{xi}, \omega_y - \Phi'_{yi}]W^*[\omega_x - \Phi'_{xk}, \omega_y - \Phi'_{yk}] = 0 \text{ for any } (\omega_x,\omega_y) \\ \text{and } i \neq k \text{ or } \parallel \nabla\Phi_i(x,y) - \nabla\Phi_k(x,y) \parallel > W_l \tag{12}$$

is satisfied; $\|\cdot\|$ denotes an appropriately defined norm in R^2. This means that cross-terms do not appear if the distance between local frequencies is greater than the maximal width of the $W(\omega_x,\omega_y)$ along the direction $\vec{\ell} = \nabla\Phi_i(x,y) - \nabla\Phi_k(x,y)$, connecting the i-th and k-th local frequency. In that case:

$$SPEC(x,y,\omega_x,\omega_y) = \sum_{i=1}^{p} \mid g_i(x,y) \mid^2 W^2[\omega_x - \Phi'_{xi}, \omega_y - \Phi'_{yi}]. \tag{13}$$

The PWD, defined by (8), may be expressed as:

$$PWD(x,y,\omega_x,\omega y) = \frac{1}{\pi^2}\int\int STFT(x,y,\omega_x+\theta_x,\omega_y+\theta_y) \\ \times STFT^*(x,y,\omega_x-\theta_x,\omega_y-\theta_y)d\theta_x d\theta_y. \tag{14}$$

From (6) and (14), appropriately adjusted to account for the class of signals defined by (10) and neglecting the artifacts, one obtains:

$$PWD(x,y,\omega_x,\omega_y) = \frac{1}{\pi^2}\sum_{i=1}^{p}\sum_{k=1}^{p} g_i(x,y)g_k(x,y)e^{j[\Phi_i(x,y)-\Phi_k(x,y)]} \\ \times \int\int W[\omega_x+\theta_x - \Phi'_{xi}, \omega_y+\theta_y - \Phi'_{yi}] \\ \times W^*[\omega_x - \theta_x - \Phi'_{xi}, \omega_y - \theta_y - \Phi'_{yi}]d\theta_x d\theta_y. \tag{15}$$

As before, the integrand in (15) is nonzero for:

$$\vec{\omega}+\vec{\theta}-\nabla\Phi_i(x,y)\in D_\omega \text{ and } \vec{\omega}-\vec{\theta}-\nabla\Phi_k(x,y)\in D_\omega \tag{16}$$

meaning that $PWD(x,y,\omega_x,\omega_y)\neq 0$, for:

$$\begin{aligned}\vec{\omega}\in D_\omega(i,k):&\ \frac{\vec{\omega}-[\nabla\Phi_i(x,y)+\nabla\Phi_k(x,y)]}{2}\in D_w\\ \text{and } \vec{\theta}\in D_\theta(i,k):&\ \frac{\vec{\theta}-[\nabla\Phi_i(x,y)-\nabla\Phi_k(x,y)]}{2})\in D_\omega\end{aligned} \tag{17}$$

where the Fourier transform $W(\omega_x,\omega_y)$ of $w(x,y)$ is assumed to be nonzero only inside a bounded region $D_w\subset R^2$, and D_w is convex and symmetric with respect to the origin (i.e. $w(x,y)$ is real), and $\vec{\omega}=(\omega_x,\omega_y)$, $\vec{\theta}=(\theta_x,\theta_y)$.

This means that the auto-terms ($i=k$) are concentrated at the local auto-frequencies of each component of signal (10), i.e. at $\vec{\omega}_i=\nabla\Phi_i(\vec{r})$, $i=1,2,\dots,p$, while the cross-terms are centered between the corresponding auto-frequencies. Relation (17) also implies that, along the axes of the two-dimensional convolution $\vec{\theta}$, all auto-terms are concentrated in the neighborhoods of $\vec{\theta}=0$. The cross-terms are dislocated from the $\vec{\theta}$ origin. Having this in mind, we conclude that the cross-terms may be removed from the PWD of a multicomponent signal, and at the same time the integration over auto-terms performed, if the convolution (17) is evaluated within a two-dimensional window function $P(\vec{\theta})$, in the following way:

$$\begin{aligned}MWD(x,y,\omega_x,\omega_y)=&\frac{1}{\pi^2}\iint P(\theta_x,\theta_y)STFT(x,y,\omega_x+\theta_x,\omega_y+\theta_y)\\ &\times STFT^*(x,y,\omega_x-\theta_x,\omega_y-\theta_y)d\theta_x d\theta_y\end{aligned} \tag{18}$$

where the region of support D_p of the window function $P(\theta_x,\theta_y)$ must comply with the conditions defined in (17), i.e. $D_p\supset D_w\equiv D_\theta(i,i)$ and $D_p\bigcap D_\theta(i,k)=\emptyset$ for $i\neq k$.

The distribution (18) is derived from the condition that its auto-terms are equal to the auto-terms in the WD. But, in contrast to the WD, this distribution is cross-terms free (under the described conditions). Note that the distribution (18) does not satisfy the marginal properties in the case of multicomponent signals. Many other distributions have been developed with the purpose of the cross-terms' reduction, [1,12,13,16]. A detailed comparison of these distributions, in the one dimensional case, may be found in [9].

Distribution (18), besides its efficiency in cross-terms removal and the preservation of the auto-terms presentation quality as in the WD, leads to a numerically more efficient method than the WD realization itself. This will be shown in Section 4 (for one-dimensional case see [4,6,7]).

5. Numerical considerations

The modified WD is compared with the conventional PWD, with respect to the number of operations needed for their respective numerical computations.

The discrete two-dimensional PWD is of the form:

$$WD(n_1,n_2,k_1,k_2) = 4\sum_{m_1=0}^{2N-1}\sum_{m_2=0}^{2N-1} x(n_1+m_1,n_2+m_2) \times x^*(n_1-m_1,n_2-m_2)e^{-j\frac{4\pi}{2N}(k_1m_1+k_2m_2)} \tag{19}$$

where N is the number of samples, determined according the sampling theorem.

The modified WD, eq.(18), may be expressed in the discrete form for a rectangular window $P_d(i_1,i_2)$, as:

$$\begin{aligned} MWD(n_1,n_2,k_1,k_2) =& \mid STFT(n_1,n_2,k_1,k_2)\mid^2 \\ &+2\sum_{i_1=0}^{L}\sum_{i_2=1}^{L} Real\{STFT(n_1,n_2,k_1+i_1,k_2+i_2) \\ &\times STFT^*(n_1,n_2,k_1-i_1,k_2-i_2)\} \\ &+2\sum_{i_1=1}^{L}\sum_{i_2=-L}^{0} Real\{STFT(n_1,n_2,k_1+i_1,k_2+i_2) \\ &\times STFT^*(n_1,n_2,k_1-i_1,k_2-i_2)\} \end{aligned} \tag{20}$$

with $L_1=L_2=L$, where $2L_1+1$ and $2L_2+1$ represent the widths of two-dimensional window $P_d(i_1,i_2)$. Sampling in the STFT is defined by sampling theorem, and so is in the modified WD due to $P_d(i_1,i_2)$.

The computation time may be reduced using an iterative procedure for computation of the STFT:

$$\begin{aligned} &STFT(n_1,n_2+1,k_1,k_2) = \{STFT(n_1,n_2,k_1,k_2)+ \\ &\mathcal{F}_{n_1}[x(n_1,n_2+N)] - \mathcal{F}_{n_1}[x(n_1,n_2)]\}e^{j\frac{2\pi}{N}k_2} \times \\ &STFT(n_1+1,n_2,k_1,k_2) = \{STFT(n_1,n_2,k_1,k_2)+ \\ &\mathcal{F}_{n_2}[x(n_1+N,n_2)] - \mathcal{F}_{n_2}[x(n_1,n_2)]\}e^{j\frac{2\pi}{N}k_1} \end{aligned} \tag{21}$$

where $\mathcal{F}_{n_1}$, F_{n_2}, are one-dimensional Fourier transforms, and the window $w(\alpha, \beta)$ is shaped as a rectangle.

Numbers of numeric operations required for the direct realization of the PWD defined by (19) (using the FFT routines), as well the numbers for the modified WD, eq.(20), are given in Table 1.

Table 1. The numbers of complex operations required for the realization of Wigner distribution and modified Wigner distribution

Method	Complex additions	Complex multiplications
Diredt WD calculation	$4N^2(\log_2 N + 1)$	$2N^2(\log_2 N + 2)$
Proposed method	$N^2(2\log_2 N + L^2 + L)$	$N^2(2\log_2 N + L^2 + 1/2)$
Proposed method with STFT iterations	$N^2(2 + L^2 + L)$	$N^2(L^2 + L + 3/2)$

Let us compare the number of multiplications needed for the computation of (19) with that in (20), taking into account (21). Savings by the proposed method are achieved if the following inequality holds:

$$2(\log_2 N + 2) > (L^2 + L + \frac{3}{2}). \tag{22}$$

To illustrate this point, take, for example, $N = 64$. Savings in the number of computations are achieved if $L \leq 3$, i.e. the width of the window $P_d(i_1, i_2)$ is less or equal to 7×7. Superiority of the proposed method is even more evident if we consider the required number of additions.

6. Numerical example

Consider the two-dimensional signal:

$$f(x, y) = \cos[20\pi(x - 0.75)^2 + 22\pi(y - 0.75)^2] + \\ 0.5e^{j[(-100\cos\frac{\pi x}{2} - 100\cos\frac{\pi y}{2})]}$$

in the range: $|x| < 0.75, |y| < 0.75$. This signal belongs to the class (10).

We have applied the Hanning window whose widths along x and y axes are $W_x = W_y = 1$. For the computation of the STFT we have taken $N = 64$ samples, while the corresponding number for the computation of the PWD is $M = 2N = 128$ samples. The STFT, the PWD and modified WD are computed at the point $(x, y) = (-0.25, -0.25)$, and the results are presented in Fig. 1.

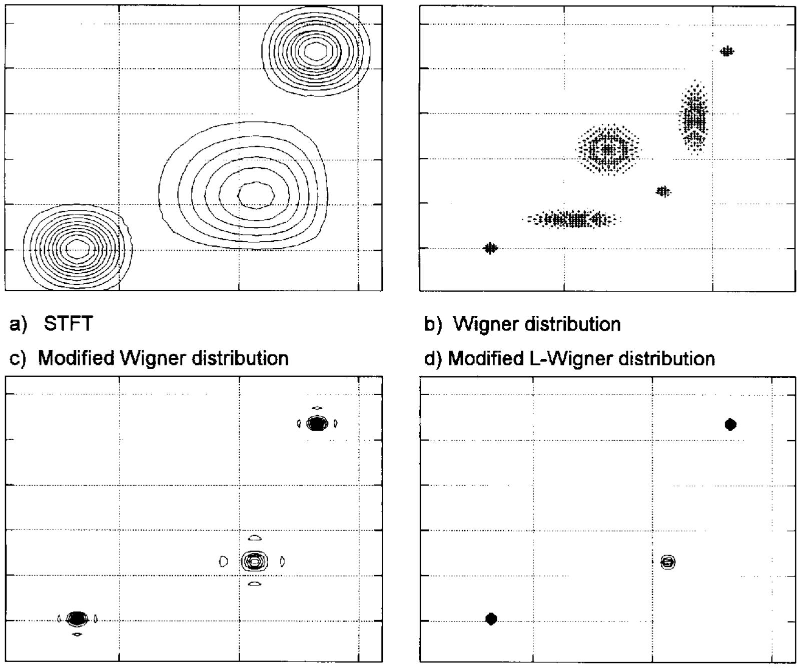

Fig. 1. Space–frequency signal representation at the point $(0, 0)$

6. Conclusion

Commonly used two-dimensional time-frequency distributions are compared with the distribution which ideally represents the local frequency. A numerically efficient method for cross-terms reduction or removal in the Wigner distribution is proposed.

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