FACTA UNIVERSITATIS (NIŠ)
SERIES: ELECTRONICS AND ENERGETICS VOL. 6, NO. 1, (1993), 41-52

AN SNR ESTIMATION ALGORITHM FOR COMPLEX BASEBAND SIGNALS USING HIGHER ORDER STATISTICS

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Abstract: In digital communication systems, the signal-to-noise ratio is an important input parameter for a variety of equalization and decoding algorithms. In typical applications such as transmission over fading channels both, current signal and current noise power are unknown. An algorithm is presented which allows estimation of signal and noise. The entire a-priori information necessary are fundamental properties of the probability density functions of signal and noise. The dependence of the quality of the estimation on the ensemble size used and on the actual signal-to-noise ratio is discussed. Simulation results for continuous as well as for blockwise transmission show that a small error of the estimate is achieved for rather small ensemble sizes.

Key words: Higher order statistic, digital communication system, signal to noise ratio, probability density function, fading channels, trellis coded modulation, PSK.

1. Introduction

In digital communication systems, the signal-to-noise ratio (SNR) is the basic quantity that determines the quality of data transmission. For instance, knowledge of the SNR at the receiver is necessary to estimate the resulting bit error rate. Moreover, quite a large number of equalization and decoding algorithms (e.g. the soft-output Viterbi decoder, cf. [1]) require the current SNR as an input parameter. In mobile radio networks, the transmitted power of each mobile should be controlled with respect to the local SNR. Of course, computation of the SNR is easy whenever either signal or noise power are known (what may be assumed for strictly time-invariant channels). However, especially the latter example illustrates that in general neither the signal power nor the noise power is known. This situation

Manuscript received July 27, 1993.

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demands for an estimation of both, signal and noise power, without any information about the current attenuation of the channel. We will call this *SNR estimation.*

In the last few years there were some attempts to solve this problem, the latest and most promising one by Shah and Hinedi [2]. They used the first and second moment of the absolute value of the considerably oversampled input signal and fed them into a rather sophisticated algorithm. There are three major shortcomings with their method: The disturbed signal must be sampled at a sampling rate much beyond the symbol rate, the accuracy of the estimation decreases with increasing intersymbol interference, and the estimation is biased. The approach presented in this paper uses simple second and fourth order statistics of a nearly arbitrary discrete-time representation of the noisy signal.

In the following, we example the complex baseband signal at the output of a synchronous demodulator and successive (whitened) matched filter (cf. fig. 1), which is sampled at symbol rate T giving a sequence $< x >$ with

$$x_\gamma = e_\gamma + n_\gamma; \quad x_\gamma, e_\gamma, n_\gamma \in \mathcal{C}; \quad \forall \gamma \in \mathcal{Z}. \tag{1}$$

γ is referred to as the discrete time, e_γ and n_γ denote the signal and noise samples respectively (fig. 1). The sequences $< e >$ and $< n >$ are assumed to be wide-sense stationary with known probability density functions $f_e(e)$ and $f_n(n)$. The task is to determine the average signal energy $\mathcal{E}\{ee^*\}$ and the average noise energy $\mathcal{E}\{nn^*\}$ purely from the observation of the sequence $< x >$.

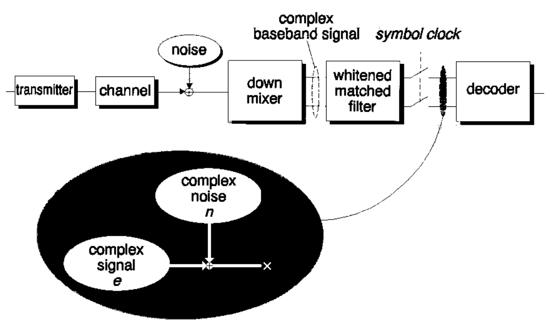

Fig. 1 Receiver structure and representation of the sampled matched-filter output by discrete-time complex stochastic processes

The rest of the paper is organized as follows: Section 2. deals with the derivation of the SNR from the sequence $< x >$, and in section 3. two examples, one for 8-PSK with additive gaussian noise, and one for two interfering 2-PSK signals are presented. Practical considerations such as the implementation of the averaging process, speed of convergence and the variance of the estimate are discussed in section 4.

2. Theory

This section is devided into three subsections: In the first subsection 2.A we give a short overview over the prerequirements which are necessary or at least convenient for the derivation of the SNR estimation algorithm in subsection 2.B. The theory section is finished by a discussion of the limitations to the application of the algorithm in subsection 2.C.

A. *Assumptions*

Real and imaginary part of a complex number z are identified by indices r and i:

$$z = z_r + jz_i \tag{2}$$

The joint probability density function (pdf) of the real and imaginary part of a discrete-time complex random process z is denoted by $f_z(z_r, z_i)$. The $2m$-th-order expected value ($2m$-th-order moment) is given by

$$\mathcal{E}\{(zz^*)^m\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(z_r^2 + z_i^2)^m f_z(z_r, z_i)dz_r dz_i. \tag{3}$$

For the following we assume that signal and noise are mutually independent complex-valued random processes with zero mean. Thus especially

$$\begin{aligned}\mathcal{E}\{en^*\} &= 0\\ \mathcal{E}\{ee^*en^*\} &= 0.\end{aligned} \tag{4}$$

Further we assume that the real and imaginary component of the noise process n be orthogonal [3]:

$$\mathcal{E}\{n_r n_i\} = \mathcal{E}\{n_r\}\mathcal{E}\{n_i\} = 0. \tag{5}$$

In the case of distorting (non-AWGN) channels it might be necessary to implement a complex whitened matched filter (i.e. four real filters) to obtain mutually independent signal and noise samples, and to fulfill condition (5).

B. *The Algorithm*

In contrast to the approach of Shah and Hinedi we compute the second and fourth order moments of the samples' absolute values. As we will see later, there is a fundamental relation between these two moments of a given distribution. The second order moments reads (in complex notation)

$$\begin{aligned}\mathcal{E}\{xx^*\} &= \mathcal{E}\{ee^* + nn^* + en^* + e^*n\} = \\ &= \mathcal{E}\{ee^*\} + \mathcal{E}\{nn^*\}\end{aligned} \tag{6}$$

and the fourth order moment:

$$\begin{aligned}\mathcal{E}\{(xx^*)^2\} = \mathcal{E}\{&(ee^*)^2 + (nn^*)^2 + (en^*)^2 + (e^*n)^2 + \\ &+ 4(ee^*nn^*) + 2(ee^*en^*) + 2(ee^*e^*n) + \\ &+ 2(nn^*en^*) + 2(nn^*e^*n)\}\end{aligned} \tag{7}$$

Now we use the required properties of e and n to eliminate or at least factorize all mixed terms in (7). Since signal and noise are mutually independent and both are zero mean processes all terms of (7) with a single appearance of n or e dissappear (cf.(4)):

$$\begin{aligned}&\mathcal{E}\{ee^*en^*\} = \mathcal{E}\{ee^*e\} \cdot \mathcal{E}\{n^*\} = 0 \diamond \mathcal{E}\{n^*\} = 0 \\ &\mathcal{E}\{ee^*e^*n\} = 0 \\ &\mathcal{E}\{nn^*en^*\} = 0 \\ &\mathcal{E}\{nn^*e^*n\} = 0\end{aligned} \tag{8}$$

Since signal and noise are independent (4),

$$\mathcal{E}\{ee^*nn^*\} = \mathcal{E}\{ee^*\} \cdot \mathcal{E}\{nn^*\}. \tag{9}$$

For the remaining square covariance term $\mathcal{E}\{(en^*)^2\}$ we obtain using (5)

$$\mathcal{E}\{(en^*)^2\} = \mathcal{E}\{e^2(n^*)^2\} = \mathcal{E}\{e^2\}\mathcal{E}\{n_r^2 - n_i^2 - 2jn_rn_i\} = 0. \tag{10}$$

Hence,

$$\mathcal{E}\{(en^*)^2\} + \mathcal{E}\{(e^*n)^2\} = 0. \tag{11}$$

For sake of a simpler notation, the following abbreviations are introduced:

$$\begin{aligned}S &:= \mathcal{E}\{ee^*\} \\ N &:= \mathcal{E}\{nn^*\} \\ E_2 &:= \mathcal{E}\{xx^*\} \\ E_4 &:= \mathcal{E}\{(xx^*)^2\},\end{aligned} \tag{12}$$

where S is the average energy of the signal constituent of the sampled input, N is the average energy of the noise constituent, and E_2 and E_4 are the second and the fourth moments of the input samples, respectively.

This notation together with the results from (8), (9) and (11) allow us to rewrite the original equations (6) and (7) in a much simpler form:

$$\begin{cases} E_2 = S + N \\ E_4 = \mathcal{E}\{(ee^*)^2\} + \mathcal{E}\{(nn^*)^2\} + 4SN. \end{cases} \tag{13}$$

Evidently,

$$\mathcal{E}\{(ee^*)^2\} = \mathcal{E}\{\mid e \mid^4\}, \quad \mathcal{E}\{(nn^*)^2\} = \mathcal{E}\{\mid n \mid^4\}. \tag{14}$$

Due to a basic law of probabilistic theory [5], the quotient of the fourth-order moment and the squared second-order moment of a probability density depends only on the shape of the density function, not on its variance:

$$k_e := \frac{\mathcal{E}\{\mid e \mid^4\}}{(\mathcal{E}\{\mid e \mid^2\})^2} \neq f(S), \quad k_n := \frac{\mathcal{E}\{\mid n \mid^4\}}{(\mathcal{E}\{\mid n \mid^2\})^2} \neq f(N) \tag{15}$$

Exploiting this property, we can rewrite (14):

$$\begin{cases} E_2 = S + N \\ E_4 = k_e S^2 + k_n N^2 + 4SN, \end{cases} \tag{16}$$

with k_e and k_n constants depending only on fundamental properties of the probability density functions of the signal and noise processes. In practice we assume (approximate) ergodicity of e and n. Hence E_2 and E_4 are determined by time-averaging the second and fourth power of the samples $< x >$, k_e and k_n are known from the employed modulation technique and the noise characteristics, respectively. The average energy S of the discrete-time signal process and the average energy N of the discrete-time noise process can then be computed by solving equation (16). Solving (16) in a straight forward manner leads to the following equations for S and N:

$$\begin{cases} S = \dfrac{E_2(2k_n - 4) \pm \sqrt{D\{x\}}}{2(k_e + k_n - 4)} \\ N = E_2 - S \end{cases} \tag{17}$$

where

$$D\{x\} := E_2^2(16 - 4k_e k_n) + 4E_4(k_e + k_n - 4). \tag{18}$$

Obviously, this solution does not necessarily exist nor is it unique. Some remarks to these problems are given in the following subsection.

C. *When does it work?*

The general solution (17) shows two problems: First, due to the square terms the solutions for S and N are not unique, and, second, the denominator $k_e + k_n - 4$ might be a zero. The latter problem can be devided into two different cases: In the first case, $k_e + k_n = 4$, but $k_e \neq k_n$. Then the square terms in S and N vanish and (16) becomes a system of two independent linear equations with a unique solution. In the second case, $k_e = k_n = 2$. This is especially true if signal and noise are gaussian processes, regardless whether one- or two-dimensional. Then all terms in S or N vanish. But this is exactly what one would expect: The superposition of two gaussian processes is a gaussian process again, and there obviously is no way at all to separate the constituent processes. This is also true for all cases where $k_x = k_e = k_n$.

Concerning the ambiguity of (17) we have made the following observation: if $k_e \neq k_n$, one solution for S is negative (drop it), one positive (take it). So signal and noise power can be determined in a unique way. If $k_e = k_n$, both solutions are positive, and we get the same pairs for signal and noise power. This can be interpreted as follows: If both processes have the same shape factor $k_e = k_n$, the algorithm cannot distinguish between signal and noise and so offers both possible combinations as solutions.

It should be clear that the above statements are based on observations, they have not yet been proved in a strict sense. But in a wide range of examined signal and noise processes, the method - except the case of two gaussian processes - never failed.

3. Examples

In this section two different examples are given. The first one examines a standard situation, 8-PSK disturbed by additive gaussian noise. At this example, we will also show how to compute the ratio of the energy per bit E_b to the one-sided spectral noise density N_0 from S and N. The second example treats two processes with identically shaped distribution functions, two interfering BPSK signals.

A. *8-PSK and White Gaussian Noise*

The first step is to determine the kurtosis of the signal and the noise process using (15). For the signal process $k_e = 1$, as for any process with a constant sample energy, and for the two-dimensional gaussian noise process

$k_n = 2$. Inserting this into (17) gives:

$$S = \sqrt{2E_2^2 - E_4} \tag{19}$$

The average energy of a noise sample is just the remaining energy

$$N = E_2 - S. \tag{20}$$

If we use the output of a matched filter at the receiver to obtain the sequence $< x_\gamma >$, we can write E_b/N_0 as

$$\frac{E_b}{N_0} = \frac{S}{3N} = \frac{\sqrt{2E_2^2 - E_4}}{3(E_2 - \sqrt{2E_2^2 - E_4})} \tag{21}$$

B. *Two Interfering BPSK Signals*

One might expect the discussed method to fail if signal and noise have identical pdf's, except for their variance. As an example we apply the algorithm to a BPSK signal which is disturbed by another BPSK signal. Figure 2 shows the signal constellations of the constituent processes e and n as well as of the resulting signal x. For sake of an easier calculation in this example we assume that both interfering BPSK signals have a constant phase offset of $\pi/4$. The numbers in figure 2 indicate the amplitudes, so the algorithm should estimate $S = 16$ and $N = 2$.

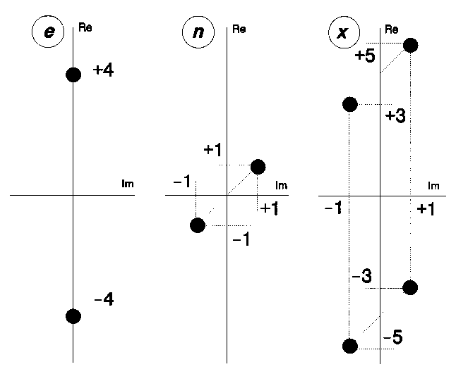

Fig. 2: Two superimposed antipodal signals (one-dimensional).

Applying (15) we find out that

$$k_e = k_n = 1. \tag{22}$$

Observing the disturbed signal x we obtain

$$\begin{aligned} E_2 &= \mathcal{E}\{\mid x \mid^2\} = \frac{(5^2+1^2)+(3^2+1^2)}{2} = 18 \\ E_4 &= \mathcal{E}\{\mid x \mid^4\} = \frac{(5^2+1^2)^2+(3^2+1^2)^2}{2} = 388 \end{aligned} \tag{23}$$

Insertion of (23) in (18) results in:

$$\sqrt{D\{x\}} = \sqrt{324(16 - 4 \cdot 1 \cdot 1) + 4 \cdot 388(1 + 1 - 4)} = 28 \tag{24}$$

and (17) gives us finally:

$$\begin{cases} S = \dfrac{18(2 \cdot 1 - 4) \pm 28}{2(1 + 1 - 4)} = \begin{cases} 2 \\ 16 \end{cases} \\ N = 18 - \begin{cases} 2 \\ 16 \end{cases} = \begin{cases} 16 \\ 2 \end{cases} \end{cases} \tag{25}$$

Here both roots of the equation for S are positive, hence the solution is not unique, as discussed in subsection 2.C.

4. Performance

For the following discussion of the performance an M-level PSK signal and Gaussian noise are assumed. However, the results are valid also for a wide range of other signal and noise characteristics. Due to the fact that the estimation algorithm uses ensemble averages of the input signal we have to distinguish between two cases: continuous transmission and blockwise transmission. In the former case the ensemble averages are approximated by continuous low-pass filtering, while in the latter time averages taken from one frame are used.

A. *Continuous Transmission*

Here the dominant parameter is the cut-off frequency of the low-pass filter used for averaging. Figure 3 shows an exemplary implementation for the

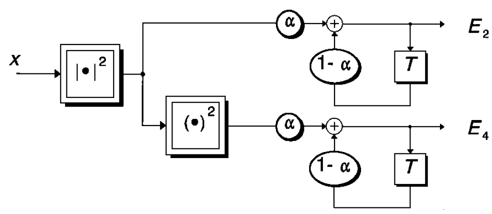

Fig. 3: A low complexity solution to compute E_2 and E_4.

computation of the second-order and fourth-order moments. The parameter α ranges from 0 to 1. A lower cut-off frequency, corresponding to a smaller parameter α in fig. 3, results in a smaller variance of the estimate, but a slower response to a change of the SNR of the input signal.

For the entire performance analysis following we assume that we are interested in the logarithmic SNR. So in the subsequent text SNR always stands for

$$SNR \doteq 10 \log_{10}(S/N) dB.$$

Fig. 4 shows the probability density function of the estimated SNR for an actual SNR of 10.0 dB and two different values of α. As expected, higher cut-off frequency (i.e. an α closer to 1) leads to a broader distribution of the estimate. There is also a slight dependence of the variance on the actual SNR which is similar to the case of blockwise transmission and will therefore be discussed in the next section. It should be emphasized that the expected value of the estimate is identical to the actual SNR, which is not always true for other known SNR algorithms (e.g. [2]).

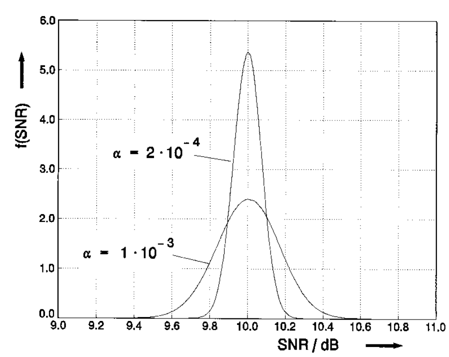

Fig. 4: Probability density function $f(SNR)$ of the estimate for two different cut-off frequencies. The parameter α is proportional to the cut-off frequency.

B. *Blockwise Transmission*

In this case the averages are calculated on basis of one frame, so there is an SNR estimate for each frame. To analyze the influence of the frame length and the actual SNR we use the mean square error (MSE) of the estimated SNR with respect to the actual SNR:

$$MSE = \mathcal{E}\{(SNR_{actual} - SNR_{estimated})^2\}$$

The MSE depends on the number of symbols in a frame, corresponding to the parameter α in the continuous case, and on the actual SNR, as briefly mentioned above. In the following discussion we consider QPSK transmission, thus one symbol corresponds to two bits. Figure 5 illustrates the influence of both parameters. Even for moderate frame lengths of 128 symbols (fig. 5: $log_2(128) = 7$) their is a wide region where MSE is less than 0.5 dB.

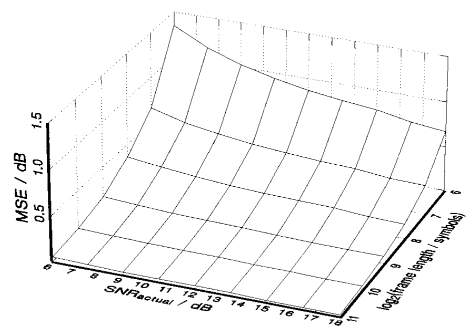

Fig. 5: MSE of the estimate as a function of frame length ans actual SNR.

The influence of the frame length on the MSE is given in more detail in fig. 6. As a rule of thumb one can say that doubling the frame length reduces the MSE by a factor of two. However, for usual values of the SNR (e.g. an SNR of 12 dB corresponds to $E_b/N_0 = 6$ dB for uncoded QPSK modulation without excess bandwidth) the MSE is much smaller than 0.5 dB even for shorter frames.

Fig. 7 shows the MSE vs. the actual SNR. Obviously, for typical values of the actual SNR the MSE of the estimate is nearly independent of the actual SNR.

Another example gives a good view of the algorithm's quality: Consider e.g. 8-PSK transmission on a non-fading channel. This means that the signal

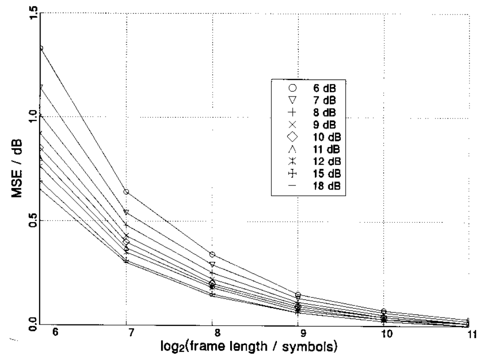

Fig. 6: Dependance of the MSE on the frame length with the actual SNR as parameter.

power can precisely be determined. Now we install a hard decision device, and compute the noise power as the mean square error (MSE) obtained as the square euclidean difference between input and output of the decision device. At an actual SNR of 9 dB, the decision-aided estimate has an MSE of 0.17 dB, while the proposed estimation algorithm ptoduces an estimate with an MSE of only 0.08 dB. Of course, this is true if and only if the signal power is exactly (!) known. In a fading environment this can not be assumed at all.

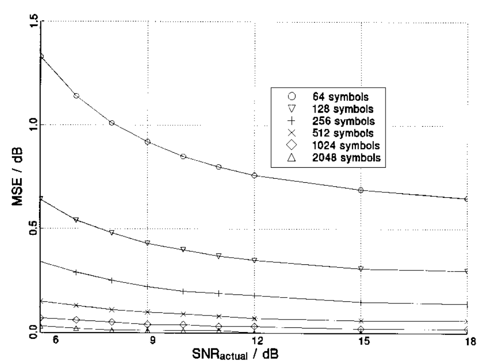

Fig. 7: The MSE of the estimate as a function of the actual SNR for several frame lengths.

5. Conclusion

In digital communication systems, the signal-to-noise ratio is an important input parameter for a variety of equalization and estimation algorithms.

While known algorithms are too complex for high data-rate applications and produce a bias error, decision aided estimation requires a-priori knowledge about the signal power, which is not available in a fading environment.

The algorithm introduced estimates the signal power as well as the signal-to-noise ratio SNR directly from a noisy signal by a simple nonlinear algorithm. The input to the algorithm can be any discrete-time representation of the receiver's input. The only a-priori information are shape factors of the signal's and noise's probability density functions. Estimates for the SNR can be obtained either continuously on a symbol-by-symbol basis or - in a block-wise transmission environment - on a frame by frame basis. The expected value of the estimate is exactly the actual SNR, while the mean square error of the estimate is significantly lower than 0.5 dB in a usual environment.

6. Additional Information

The proposed algorithm has successfully been implemented in a prototype transmitter/ receiver for several trellis-coded modulation (TCM) schemes, namely trellis-coded 8-PSK and quaternary partial-response continuous phase modulation (CPM) with several complexity reduction strategies. Due to the high accuracy of the algorithm it has been possible to measure differences in the performance of various complexity reduction schemes of less than 0.1 dB.

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