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SYSTEMS FOR HOMOGENEOUS ELECTROSTATIC FIELD GENERATION USING CHEBYSHEV POLYNOMIALS *

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Abstract. *We propose an optimization method for homogeneous electrostatic field generation. The approach is based on the expansion of axial potential in Chebyshev polynomials for circular symmetrical electrodes arrangement. Dimensions of the electrodes are chosen so that, in the region within the system, a homogeneous electrostatic field exists.*

Key words: *Homogeneous electrostatic field, Chebyshev polynomials Primary cell of the first order, Primary cell of the second order*

1. INTRODUCTION

Uniform magnetic field generation is often based on the well-known Helmholtz coil composed of two circular coils that produce maximum magnetic field uniformity near the centre of the system when the distance between the coils is equal to the radius of the coils [1]. Numerous improvements on the Helmholtz design have been done in order to produce larger volumes of space with homogeneous magnetic fields. This case has been extended to an arbitrary number of circular coils on a spherical surface [2-3] and square coils placed on the same cylindrical surface [4].

Homogenously electric field generation problems are less resolved than the previous one. Two charged rings produce the best electric field uniformity near the centre of the system when the distance between the rings is $\sqrt{6}$ times greater than the ring radius [5]. A spherical conducting surface at defined potentials can be used to create the uniform electric field near the centre of the sphere [6].

In previously published papers [7, 8], the authors have suggested a complex coil system for uniform electric field generation. Modelling and optimization process [9] of the system is based on the system which consists of two thin circular electrodes of absolutely equal but

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opposite potential values. Better results could be achieved using a large number of circular electrodes pairs. Very good results are obtained using biconical electrodes [8] and [10].

The charged circular electrodes placed on a spherical surface are considered in the paper in order to produce a homogeneous electrostatic field near the centre of the sphere. As a basic element the so-called primary cell is used. The system is designed using an expansion of the axial potential distribution into a Chebyshev series. The system dimensions are chosen to eliminate the subsequent terms different than linear, as many as possible. In this paper the results for primary cells of the first and second kind are presented.

2. Outline of the Problem

By analogy to Helmholtz coils that produce a uniform magnetic field, it is possible to produce a homogeneous electric field from two circular electrodes. The system presented in Fig. 1, the so-called primary cell, is considered. It consists of a pair of flat coaxial thin charged circular electrodes of radius d and separation distance $2h$. The electrodes are on the absolutely equal and opposite potential values, so that the potential of the symmetrical plane is zero. Q is the charge of circular electrode of potential U, $-Q$ is the charge of circular electrode of potential $-U$.

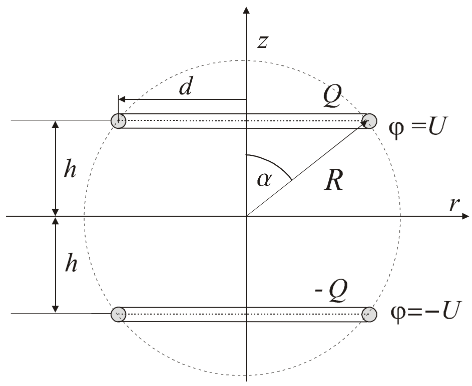

Fig. 1 Two circular electrodes system for homogeneous electric field generation

For the electrode system shown in Fig. 1, the potential distribution along the system axis, $r = 0$, is

$$\varphi(z) = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{\sqrt{d^2+(z-h)^2}} - \frac{1}{\sqrt{d^2+(z+h)^2}}\right] \tag{1}$$

where r and z are cylindrical coordinates.

Substituting $d = R\sin\alpha$, $h = R\cos\alpha$ and $\xi = z/R$, Fig. 1, in the previous expression, the following form is obtained

$$\varphi(\xi) = \frac{Q}{4\pi\varepsilon R}\left[\frac{1}{\sqrt{1+\xi^2-2\xi\cos\alpha}} - \frac{1}{\sqrt{1+\xi^2+2\xi\cos\alpha}}\right], \tag{2}$$

R is electrode distance from the origin, $R = \sqrt{d^2 + h^2}$.

When the origin lies in a plane of symmetry of the system, the potential has corresponding symmetry and in the area $|z| < R$, the function $\varphi(\xi)$ can be expanded into an infinite decreasing power series,

$$\varphi(\xi) = \sum_{n=0}^{\infty} \varphi_{2n+1}\xi^{2n+1} , \tag{3}$$

with constant coefficients, which contains only odd powers of axial variable and depends on the system dimension.

The components of the electrostatic field can be found by

$$\mathbf{E} = -\,\mathrm{grad}\,\varphi . \tag{4}$$

In order to provide a homogenous electrical field, the potential has to be linearly dependent on the distance, along the axis. Since the coefficients depend on electrodes positions, it is possible to choose primary cell positions to eliminate as many as possible subsequent terms different than linear. The homogeneity criterion is the number of vanishing terms in a series.

In the general case, by equating the coefficient of ξ^{2N+1} to zero value in the expression (3), the following relation is obtained

$$\varphi_{2N+1} = 0 \text{ for } N \geq 1. \tag{5}$$

This equation has N real positive roots, where $n = 1, 2,\ldots, N$, and each of these solutions defines one degenerative primary cell with parameters d_n and h_n. N defines primary cell order. All the primary cells of the same order, but different degeneration grade, should be placed on a spherical surface of the radius R, $R^2 = d_n^2 + h_n^2$.

In order to eliminate the influence of other series terms from z^3 to z^{2N-1}, the following system of equations has to be satisfied,

$$\varphi_{2n+1} = 0 \text{, for } n = 1,2,...,N-1 . \tag{6}$$

The charges of electrodes, Q_n, are obtained solving this system of equations.

1.1 Taylor polynomial expansion

The system presented in Fig. 1 is considered. Since the origin lies in a plane of symmetry of the system, the function $\varphi(\xi)$, within the volume enclosed by the coils, may be expressed in Taylor series as

$$\varphi(\xi) = \frac{Q}{4\pi\varepsilon R}\sum_{n=0}^{\infty} \xi^n [P_n(\cos\alpha) - P_n(-\cos\alpha)] \tag{7}$$

and written in the form (3) where [8]

$$\varphi_{2n+1} = \frac{Q}{2\pi\varepsilon R} P_{2n+1}(\cos\alpha)\,, \tag{8}$$

where $P_n(x)$ is Legendre polynomial of the first kind.

Because the series (7) is decreasing, the field in the area $|z| < R$ is homogeneous when the system dimensions are chosen so that the linear term is dominant in the central area of the system. By annulling the coefficient of ξ^3, the equation $\varphi_3 = 0$ is obtained. By solving this equation, the dimensions of the primary cell of the first order, N=1, are obtained, Fig. 1,

$$h = 0.7745966692\,R \text{ and } d = 0.63245552\,R.$$

By setting the coefficient φ_5 to zero value, the primary cell of the second order is obtained, N=2, Fig. 2, whose dimensions are

$$d_1 = 0.422892524R\,,\ h_1 = 0.906179846\,R \text{ and}$$
$$d_2 = 0.842654122\,R\,,\ h_2 = 0.538469310\,R\,.$$

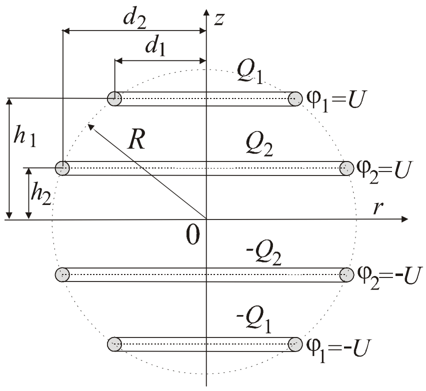

Fig. 2 Four circular electrodes system for homogeneous electric field generation

1.2 Chebyshev polynomial expansion

Also, the aim of this paper is to design a system for generating a homogeneous electrostatic field using an expansion of the potential distribution along the system axis into a Chebyshev series. The system dimensions are chosen so that only linear term remains.

The function $\varphi(\xi)$ can be expressed in the following form

$$\varphi(\xi) = \sum_{n=0}^{\infty} a_{2n+1} T_{2n+1}(\xi), \tag{9}$$

where $T_n(\xi)$is Chebyshev polynomials of the first kind [11-13]. A few first Chebyshev polynomials are:

$$T_1 = x$$

$$T_3 = 4x^3 - 3x$$

$$T_5 = 16x^5 - 20x^3 + 5x$$

$$\vdots$$

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} 2^{n-2k-1} x^{n-2k} .$$

They are orthogonal with a weight function $1/\sqrt{1-x^2}$ in the interval [−1;1],

$$\int_{-1}^{1} \frac{T_n(\xi)\, T_s(\xi)}{\sqrt{1-\xi^2}} \mathrm{d}\,\xi = \begin{cases} 0, & n \neq s \\ \pi/2, & n = s \neq 0 \\ \pi, & n = s = 0 \end{cases} \tag{10}$$

so the coefficients in series (9) can be obtained using

$$a_{2n+1} = \frac{Q}{\pi^2 \varepsilon R} \int_0^1 \frac{T_{2n+1}(\xi)\varphi(\xi)}{\sqrt{1-\xi^2}} \mathrm{d}\,\xi .$$

Substituting function $\varphi(\xi)$ with (7), the above expression becomes

$$a_{2n+1} = \frac{8}{\pi} \sum_{m=0}^{\infty} P_{2m+1}(\cos\alpha) \int_0^1 \frac{T_{2n+1}(\xi)}{\sqrt{1-\xi^2}} \xi^{2m+1}\, \mathrm{d}\,\xi . \tag{11}$$

First several coefficients are:

$$a_1 = \frac{Q}{\pi^2 \varepsilon R} \sum_{m=0}^{\infty} \mathrm{P}_{2m+1}(\cos\alpha) \frac{(2m+1)!!}{(2m+2)!!}$$

$$a_3 = \frac{Q}{\pi^2 \varepsilon R} \sum_{m=0}^{\infty} P_{2m+1}(\cos\alpha) \frac{(2m+1)!!}{(2m+2)!!} \frac{m}{m+2}$$

$$\cdots$$

$$a_{2n+1} = \frac{Q}{\pi^2 \varepsilon R} \sum_{m=0}^{\infty} P_{2m+1}(\cos\alpha) \frac{(2m+1)!!}{(2m+2)!!} \frac{m(m-1)(m-2)...(m-n+1)}{(m+2)(m+3)(m+4)...(m+n+1)}$$

for $n \geq 1$.

Finally, the function (9) can be written in the following form:

$$\varphi(\xi) = \sum_{n=0}^{\infty} \frac{Q}{\pi \varepsilon R} \sum_{m=0}^{\infty} P_{2m+1}(\cos\alpha) \frac{(2m+1)!!}{(2m+2)!!} \frac{m(m-1)...(m-n+1)}{(m+2)(m+3)...(m+n+1)} \times$$
$$\sum_{k=0}^{\left[\frac{2n+1}{2}\right]} (-1)^k \frac{2n+1}{2n-k+1} \binom{2n-k+1}{k} 2^{2n-2k} \xi^{2n-2k+1} \tag{12}$$

i.e. in the short form,

$$\varphi(\xi) = \sum_{n=0}^{\infty} b_{2n+1}\xi^{2n+1}, \tag{13}$$

where

$$\begin{aligned} b_1 &= A_1 - 3A_3 + 5A_5 - 7A_7 + 9A_9 - 11A_{11} + 13A_{13} - \dots \\ b_3 &= 4A_3 - 20A_5 + 56A_7 - 120A_9 + 220A_{11} - 13A_{13} + \dots \\ b_5 &= 16A_5 - 112A_7 + 423A_9 - 1232A_{11} + 2912A_{13} - \dots \\ b_7 &= 64A_7 - 576A_9 + 2816A_{11} - \dots \\ &\vdots \end{aligned} \tag{14}$$

Since the series (13) is decreasing, by setting the coefficient b_3 of ξ^3 to zero value, the first order primary cell dimensions are obtained, Fig. 1. The results depend on the terms number, defined by the coefficient b_3. The values for Chebyshev cells dimensions are approaching the Taylor's cells dimensions, when increasing accuracy during calculation of Chebyshev cells dimensions.

The function $\varphi(\xi)$ along the system axis is now approximately linear function of z

$$\varphi(\xi) = b_1\xi + b_5\xi^5 + b_7\xi^7 + \cdots \approx b_1\xi \tag{15}$$

and the electric field is homogeneous and approximately equal to the electric field value in the origin

$$E \approx E_0 = b_1 / R\ . \tag{16}$$

The field within the coils is given by constant term b_1/R plus a series of error terms which can be neglected with respect to linear term. The greater the primary cell order, the smaller the error.

In the case when the coefficients of ξ^5 is equal to zero in (13), the primary cell dimensions of the second order is obtained, Fig. 2. The electrodes positions are determined using the condition $b_5 = 0$.

Table 1 The system dimensions dependence on terms number in the expression for coefficients calculation b_3, and b_5, respectively

Number of members	Primary cell of the first order	Primary cell of the second order	
	h/R	h_1/R	h_2/R
1	0.9328	0.9885	0.7181
2	0.8225	0.9322	0.6220
3	0.7694	0.8779	0.5689
4	0.76119	0.9297	0.5445
5	0.77260	0.9489	0.5370
6	0.77594	0.9104	0.5369
Taylor series	0.77459	0.9062	0.5384

Table 1 shows the primary cells dimensions of the first and second order for different number of terms in expression (14).

3. Numerical Results

In accordance with the analysis presented above, a general numerical program has been developed using the software Mathematica 7.0 and modelled using the software Femm, [14]. In Figs. 3 and 4, the uniformities of the fields in the plane $z = 0$ for the two primary cells are compared. A homogeneity factor ε, which represents the deviation of the field at a position (r, z) with respect to the field in the centre (0, 0), can be introduced:

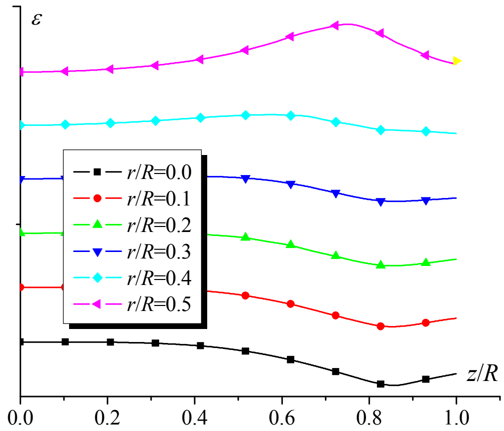

Fig. 3 Uniformity of the axial electric field in the plane $z = 0$ for the primary cell of the first order

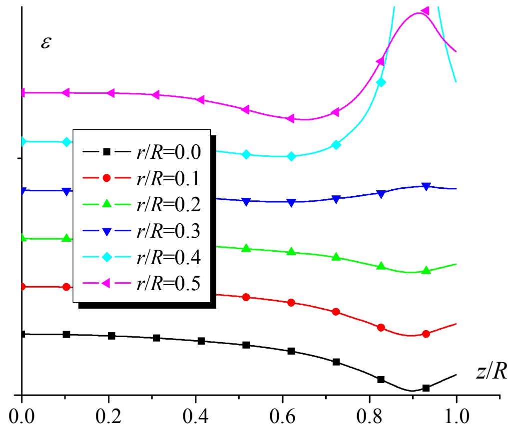

Fig. 4 Uniformity of the axial electric field in the plane $z = 0$ for the primary cell of the second order

$$\varepsilon = [E(r,z) - E(0,0)] / E(0,0) . \tag{16}$$

The fields are calculated along the lines parallel to the z axis, where $r = const$. Normalized values of r are given in the pictures. Each curve has its own zero ordinate. They are placed on equal distances along the z axis [15].

Equipotential lines for primary cells of the first and second order are presented in Fig. 5 and Fig. 6, respectively. Since the electric field is symmetric, only the region of $r > 0$ is shown. All lines are plotted using software Femm [14].

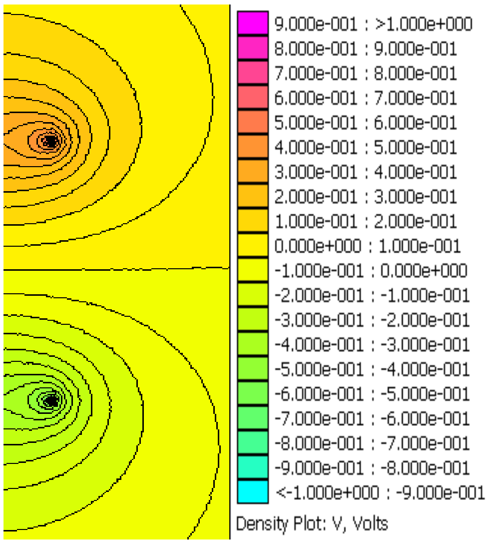

Fig. 5 The equipotential lines for the primary cell of the first order

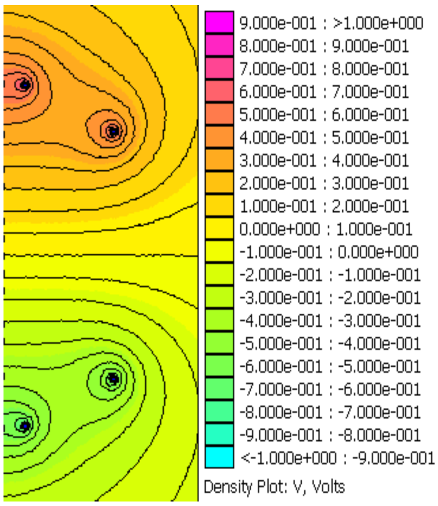

Fig. 6 The equipotential lines for the primary cell of the second order

3. Conclusion

In this paper the new system for homogeneous electric field generation is observed. As a basic element the primary cell is used. In order to obtain a homogeneity electric field, two different approaches are considered. The first one uses the axial potential function expansion into the Taylor series and the second one uses the expansion into the Chebyshev series.

The investigations show that both approaches define primary cells of the same dimensions. Solving the problem using the Chebyshev expansion is complex from the mathematical point of view compared to Taylor's expansion which has the advantage in practical applications.

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SISTEMI ZA GENERISANJE HOMOGENOG ELEKTROSTATIČKOG POLJA PRIMENOM ČEBIŠEVLJEVIH POLINOMA

Zlata Z. Cvetković, Mirjana T. Perić, Ana N. Vučković

U radu je predložen optimizacioni metod za generisanje homogenog elektrostatičkog polja. On se zasniva na razvoju izraza za aksijalni potencijal u Čebiševljeve polinome kada su torusne elektrode sistema raspoređene oko sfere. Dimenzije elektroda su određene tako da je u unutrašnjosti sistema dobijeno homogeno elektrostatičko polje.

Ključne reči: *Homogeno electrostatičko polje, Čebiševljevi polinomi, Osnovna ćelija prvog reda, Osnovna ćelija drugog reda*