FACTA UNIVERSITATIS
Series: **Automatic Control and Robotics** Vol. 8, N° 1, 2009, pp. 137 - 147

CONTRIBUTION TO NEW INTERPRETATION OF THE EULER-BERNOULLI EQUATION

UDC 531.39 519.87 624.0449

Mirjana Filipović[1], Vladimir Kokotović[2]

[1]Mihajlo Pupin Institute, Belgrade, Serbia
E-mail: mira@robot.imp.bg.ac.rs
[2]Ford Motor Company, Research Innovation Center, Michigan
E-mail: vkokotov@ford.com

Abstract. *The introduction of link flexibility into the mathematical model complex elastic system of robots that are in contact with dynamic environment is considered. Special attention is devoted to the kinematics and dynamics of movement of elastic link in the frame of robotics configuration. The Euler-Bernoulli equation and its solution should also be expanded according to the requirements of the motion complexity of robotic systems. The elastic deformation by both amplitude and frequency is a dynamic value which depends on the total dynamics of the robot system movements. The stiffness (and damping) matrix is a full matrix. Mathematical model of the actuators also comprises coupling between elasticity forces. By superposing the particular solution of oscillatory nature (integral of Daniel Bernoulli), and the stationary solution of forced nature, we can obtain any flexible deformation of a considered mode. General form of the elastic line is a direct outcome of the system motion dynamics.*

Key words: *Modeling, robot, elastic deformation, coupling, kinematics, dynamics*

1. INTRODUCTION

Modeling and control of elastic robotic systems has been a challenge to researchers in the last four decades. In [1], the feedback control was formed for the robot with flexible links (two-beam, two-joint systems) with distributed flexibility, robots with flexible links being also dealt with in [13]. Authors [21] derived dynamic equations of the joint angle, the vibration of the flexible arm, and the contact force. The paper [27] presents an approach to end point control of elastic manipulators based on the nonlinear predictive control theory. The paper [26] presents method for the generation of efficient kinematics and dynamic models of flexible robots.

Received March 10, 2010

Work [18] presents the derivation of the equations of motion for application mechanical manipulators with flexible links. In [19] the equations are derived using Hamilton's principle, and are nonlinear integro-differential equations. Method of separation of variables and the Galerkin's approach are suggested in paper [20] for the boundary-value problem with time-dependent boundary condition.

Mathematical model of a mechanism with one DOF (degree of freedom), with one elastic gear was defined by Spong [24], [25]. Based on the same principle, elasticity of gears is introduced into the mathematical model in this paper, as in papers [8]-[12]. However, as far as the introduction of link flexibility into the mathematical model is concerned, it is necessary to point out to some essential problems in this domain.

We consider that EBA ("Euler-Bernoulli approach") (used in [6] etc.) and LMA ("Lumped-mass approach") (used in [3]), are two comparable methods addressing the same problem but from different aspects [8], [9], [11], [12]. As the equation of motion for the mode tip point is essentially LMA and it follows directly from the equation of flexible line obtained via EBA for the preset boundary conditions, it clearly comes out that the structures of these equations are the same (whereas the content of elements of these structures is not the same).

In the previous literature [4] - [6], [14], [15], [17], [22], [26], [27] the general solution of the motion of an elastic robotic system has been obtained by considering flexible deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli. We consider that any elastic deformation can be presented by superimposing D. Bernoulli particular solutions of the oscillatory character and stationary solution of the forced character. See papers [8], [9], [11], [12].

Elastic deformation (of flexible links and elastic gears) is a quantity which is at least partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (both of stiffness and damping) are "known", at least partly and at that level can be included into the process of defining the reference motion. As far as the working regime of the robot is concerned we think that all forces should participate in generating elastic deformations and that it is a crude approximation to assume that elastic effects are generated only by gravitational force, or only by the environment force as in [21], or that Coriolis and centrifugal forces can be neglected altogether that elastic deviations are so small that inertia matrix is not dependent on them, as assumed in [16].

In our paper we do not use "assumed modes technique" proposed by Meirovitch [22]. In our paper we form Euler Bernoulli equation but we do not use "assumed modes technique" in contrast to our contemporaries. Elastic deformation is a consequence of the overall dynamics motion of the robotic system, in our opinion. Let us emphasize once again that in this paper we propose a mathematical model solution that includes in its root the possibility for analyzing simultaneously both present phenomena – the elasticity of gears and the flexibility of links, and the idea originated from [2], but now on the new principles. We show how the continuously present environment dynamics force affects the behavior of an elastic robot system.

Our future work: The mechanism would be modelled to contain elastic elements and to generate vibrations, which are used for conveying particulate and granular materials in [7].

In Section 2 we demonstrate that the elastic deformation is a consequence of the overall dynamics of the robotic system which is essentially different from the method that was used until today and includes the usage of "assumed modes technique". Procedure of defining the dynamic model with all elements of coupling is presented completely as well as with dynamic effects of present forces defined in Section 3. Kinematic model of system is created in Section 4. Section 5 analyzes simulational example for movement dynamic of an elastic robotic pair with elastic gear and flexible link (two modes). Section 6 gives some concluding remarks.

2. Dynamics of Change of Elastic Deformations

With the intensive development of the new technical areas such as robotics especially strengthened by the development of the data computing process, demanded and enabled that elastic deformation was considered really as the dynamic value which depended on the system parameters. The elastic deformation is a dynamic value by both amplitude and frequency and it is the result of the total system movements i.e. outer and inner, dynamic and static forces. Such elastic deformation should exist in the dynamics of the robot system movements. The synthesis of the robot system dynamics should be processed on the basis of the completely new different principles comparing to [22], with models based on the known, classic dynamics, the elasticity theory and the oscillation theory, where the elastic deformations are described as dynamic values of the inner and outer load which influence the total dynamics of the robot system movements.

The elastic deformation exists even in the state of inaction and then it depends on the gravity forces i.e. mechanism configuration. That means that the elastic deformation depends on the robot system characteristics and it can be calculated in any chosen moment. When we conduct the mechanism through the reference trajectory, the elastic deformation also exists but now at the reference level without the influence of the disturbance.

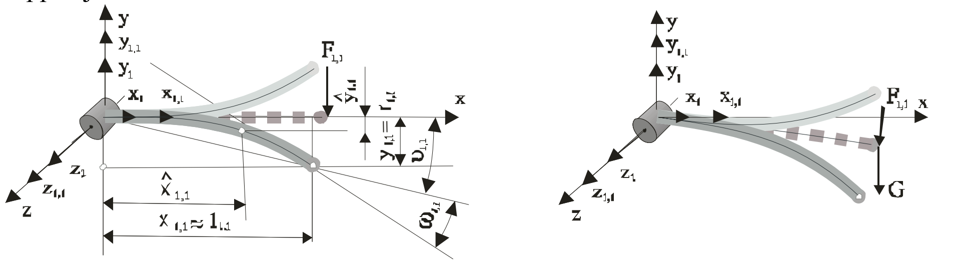
The elastic deformation cannot be defined in advance (with both amplitude and frequency) and put in the system but completely inversely. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system movements. That means that the elastic deformation amplitude and its frequency change depending on the forces (inertial forces, Coriolis, centrifugal forces, gravity forces as well as coupling forces between the present modes, and the play of the environment forces). It, of course, depends on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor movements, etc.

3. Dynamics

The Euler-Bernoulli equation was written in 1750. It was written by Bernoulli, a physicist and Euler, a mathematician, his long-time friend and colleague. They did not even dream about the robotics and the knowledge we now have at our disposal. But, although it was made more than 250 years ago, the Euler-Bernoulli equation is still valid and it can be connected logically to contemporary knowledge in robotics. Equation of the elastic line of beam bending has the following form (as defined by Euler and Bernoulli):

$$\hat{M}_{1,1} + \hat{\varepsilon}_{1,1} = 0, \qquad \hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^{\;2}} \tag{1}$$

Equation (1) was defined under the assumption that the elasticity force is opposed only by the inertial force proper. Besides, it is supposed by the definition that the motion in (1) is caused by an external force $F_{1,1}$, suddenly added and then removed. Bernoulli presumed the horizontal position of the observed body as its stationary state (in this case it matches the position x - axis, see Fig. 1 a). With such a presumption, the oscillations happen just around the x - axis.



a) idealized motion according D. Bernoulli b) motion in case of presence gravity force

Fig. 1 Motion of elastic body

If Bernoulli, in any case, had included the gravity force G in (1), the situation would have been more real. Then the stationery body position would not have matched the x - axis position, but the body position would have been little lower and the oscillations would have happened around the new stationery position as presented in Fig. 1 b.

The Euler-Bernoulli equation (1) expanded in [8], [9], [11], [12] from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms.

By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

The motion of the considered robotic system mode is far more complex, Fig. 1 b, than the motion of the body presented in Fig. 1 a. This means that the equations that describe the robotic system (its modes) must also be more complex than (1) formulated by Euler and Bernoulli. This fact is overlooked, and the original equations are widely used in literature to describe the robotic system motion. This is very inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost. Hence, the necessity of expanding the source equations for the purpose of modeling robotic systems should be especially emphasized, and this should be done in the following way:

- based on the known laws of dynamics, (1) is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode. It is assumed that the forces of coupling (inertial, Coriolis, and elastic) between the present modes are also involved, which yields structural difference between (1) in the whole model.
- Damping is an omnipresent flexibility characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 (\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}^2_{1,1}} = 0 \qquad (2)$$

$\eta_{1,1}$ is a factor characterizing the share of damping in the total flexibility characteristic.

The load moment is composed of all forces acting on the first mode of the link and these are inertial forces (own and coupled forces), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the force of the environment dynamics, which is via Jacobian matrix transferred to the motion of the first mode that come out directly from the motion dynamics of elastic bodies. They become more complex. This means that all these forces participate in generating bending moment i.e. in forming elastic deformation as well as the elasticity line of the first mode. In that case the model of elastic line of the first mode of the elastic link has the form of the Euler-Bernoulli equation:

$$\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_{1,1}+j_{1,1}^{T}F_{uk}+z_{1,j}\cdot\varepsilon_1+\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}_{1,1}^2}=0 \tag{3}$$

Let us consider a robotic system with m links, whereby the first link contains n_1 modes, the second link n_2 modes, etc. the m^{th} link contains n_m modes. Model of the elastic line of this complex elastic robotic system is given in the matrix form by the following Euler-Bernoulli equation:

$$\hat{H}\cdot\frac{d^2\hat{y}}{dt^2}+\hat{h}+j_e^{T}\cdot F_{uk}+z\cdot\Theta\cdot\varepsilon+\hat{\varepsilon}=0 \tag{4}$$

Robotics researchers are especially interested in the motion of the first mode tip.

The equation of motion of the forces involved at any point of the elastic line of the first mode, including the point of the first mode tip, can be defined from the Euler-Bernoulli equation (3). The equation of motion of all forces at the first mode tip for the given boundary conditions can be defined by the following equation:

$$H_{1,j}\frac{d^2y_{1,j}}{dt^2}+h_{1,1}+j_{e1,1}{}^{T}\cdot F_{uk}{}^{T}+z_{1,j}\cdot\varepsilon_{1,j}+\varepsilon_{1,1}=0\,\left|\begin{array}{l}\Sigma F=0\,(\Sigma M=0)\\ \text{at the point of}\\ \text{first mode tip}\end{array}\right. \tag{5}$$

Equation (5) is interesting because it allows one to calculate the position of the first mode tip. If we know the position of each mode tip, we can always calculate the position of the link tip too and eventually the position of the robot tip.

The equation of motion of all the forces at the point of each mode tip of any link can be defined from the Euler-Bernoulli equation (4) by setting the boundary conditions. Vector equation of motion of all the forces involved for each mode tip of any link is:

$$H\frac{d^2y}{dt^2}+h+j_e^{T}\cdot F_{uk}+z\cdot\Theta\cdot\varepsilon+\varepsilon=0\,\left|\begin{array}{l}\Sigma F=0\,(\Sigma M=0)\\ \text{at the tip of}\\ \text{any mode of the}\\ \text{link considered}\end{array}\right. \tag{6}$$

Mathematical model of all m motors can be written in a vector form as:

$$u=R\cdot i+C_E\cdot\dot{\theta},\quad C_M\cdot i=I\cdot\ddot{\theta}+B_u\cdot\dot{\theta}-S\cdot(z_m\cdot\varepsilon+\varepsilon_m)\,\left|\begin{array}{l}\Sigma M=0\\ \text{about the rotation axis}\\ \text{of the each motors}\end{array}\right. \tag{7}$$

4. KINEMATICS

First we can analyse the solution of the Euler-Bernoulli equation (1). General solution of motion, i.e. the form of transversal oscillations of flexible beams can be found in the method of particular integrals of D. Bernoulli, that is:

$$\hat{y}_{to1,1}(\hat{x}_{1,1},t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{to1,1}(t) \tag{8}$$

See Fig. 1 a. By superimposing the particular solutions (8), any transversal oscillation can be presented in the following form:

$$\hat{y}_{to1}(\hat{x}_{1,j},t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{to1,j}(t) \tag{9}$$

As already mentioned, equations (1), (8), (9) are defined under the assumption that the elasticity force is opposed only by the inertial force proper. The solution (8), (9) of D. Bernoulli satisfies these assumptions.

The Bernoulli solution (8), (9) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. Daniel Bernoulli solution (8), (9) should be expanded.

Hence, we should especially emphasize the necessity of expanding the source solution (8) with the stationary character of the elastic deformation caused by the forces involved.

By superposing the particular solution of oscillatory nature, and the stationary solution of forced nature, any flexible deformation of a considered mode may be presented in the following general form:

$$\hat{y}_{1,1} = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot (\hat{T}_{st1,1}(t) + \hat{T}_{to1,1}(t)) \tag{10}$$

$\hat{T}_{st1,1}$ is the stationary part of flexible deformation caused by stationary forces that vary continuously over time. $\hat{T}_{to1,1}$ is the oscillatory part of flexible deformation as in (8).

Component $\hat{X}_{1,1}(\hat{x}_{1,1})$ describes a possible geometrical relation between $\hat{y}_{1,1}$ and $\hat{x}_{1,1}$. Component $\hat{T}_{st1,1} + \hat{T}_{to1,1}$ describes the dependence of flexure $\hat{y}_{1,1}$ on flexibility force, which is the only time-varying quantity in expression (10). By superposing solutions (10), any flexible deformations of a flexible link with an infinite number of degrees of freedom may be presented in the following form:

$$\hat{y}_{1}(\hat{x}_{1,j},t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot (\hat{T}_{st1,j}(t) + \hat{T}_{to1,j}(t)) \tag{11}$$

By superimposing the particular solutions of oscillatory character and stationary solution of forced character, position and orientation of any elastic deformation of one mode can be presented in the following basic form (solution of (3)):

$$\hat{y}_{1,1} = \hat{a}_{1,1}(\hat{x}_{1,1}, \hat{T}_{st1,1}, \hat{T}_{to1,1}, t), \qquad \hat{\psi}_{1,1} = \hat{d}_{1,1}(\hat{x}_{1,1}, \hat{T}_{st1,1}, \hat{T}_{to1,1}, t) \tag{12}$$

Solution of the system (4) and dynamic motor motion, i.e. the form of its elastic line, can be obtained in the presence of the dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration.

$$\begin{aligned}&\hat{y}=\hat{a}(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{toi,j},\overline{\theta},\alpha,t),\ \hat{x}=\hat{b}(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{toi,j},\overline{\theta},\alpha,t),\ \hat{z}=\hat{c}(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{sti,j},\overline{\theta},\alpha,t)\\&\hat{\psi}=\hat{d}(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{toi,j},\overline{\theta},\alpha,t),\ \hat{\xi}=\hat{e}(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{toi,j},\overline{\theta},\alpha,t),\ \hat{\varphi}=f(\hat{x}_{i,j},\hat{T}_{sti,j},\hat{T}_{toi,j},\overline{\theta},\alpha,t)\end{aligned} \tag{13}$$

Thus we defined the position and orientation of each point of the elastic line in the space of Cartesian coordinates. It should be pointed out that the form of elastic line comes out directly from the dynamics of the system motion.

The motion of the mode tip, its position and orientation, are defined by the sum of the stationary and oscillatory motion (solution of (5)).

$$y_{1,1}=a_{1,1}(x_{1,1},T_{st1,1},T_{to1,1},t),\ \psi_{1,1}=d_{1,1}(x_{1,1},T_{st1,1},T_{to1,1},t) \tag{14}$$

The robot tip motion is defined by the sum of the stationary and oscillatory motion of each mode tip plus the dynamics of motion of the motor powering each link, as well as by the included robot configuration (solution of (6) and (7)):

$$\begin{aligned}&y=a(x_{i,j},T_{sti,j},T_{toi,j},\overline{\theta},\alpha,t),\ x=b(x_{i,j},T_{sti,j},T_{toi,j},\overline{\theta},\alpha,t),\ z=c(x_{i,j},T_{sti,j},T_{sti,j},\overline{\theta},\alpha,t)\\&\psi=d(x_{i,j},T_{sti,j},T_{toi,j},\overline{\theta},\alpha,t),\ \xi=e(x_{i,j},T_{sti,j},T_{toi,j},\overline{\theta},\alpha,t),\ \varphi=f(x_{i,j},T_{sti,j},T_{toi,j},\overline{\theta},\alpha,t)\end{aligned} \tag{15}$$

From (15) we can calculate the motion of each mode tip and link, and finally, of the robot tip motion.

5. Example

Robot starts from point "A" (Fig. 2) and moves toward point "B" in the predicted time $T=2[s]$. Dynamics of the environment force [23] is included into the dynamics of system's motion.

The adopted velocity profile is trapezoidal ($\dot{q}^{o}_{\max}=0.9817(rad/s)$, with the acceleration/deceleration period of $0.2\cdot T$ ($\ddot{q}^{o}_{\max}=\pm 2.4544(rad/s^2)$.

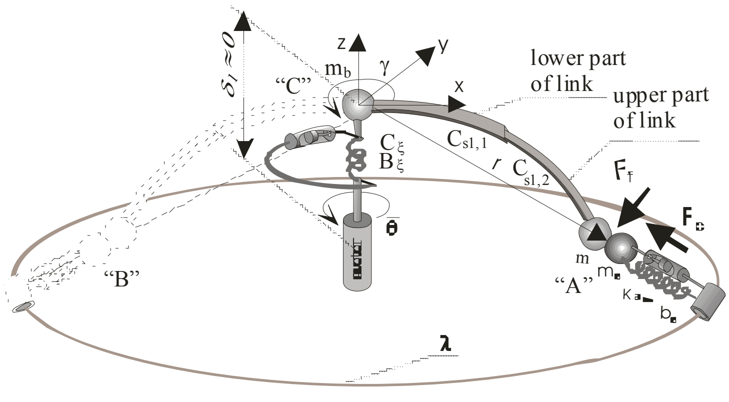


Fig. 2 The robotic mechanism

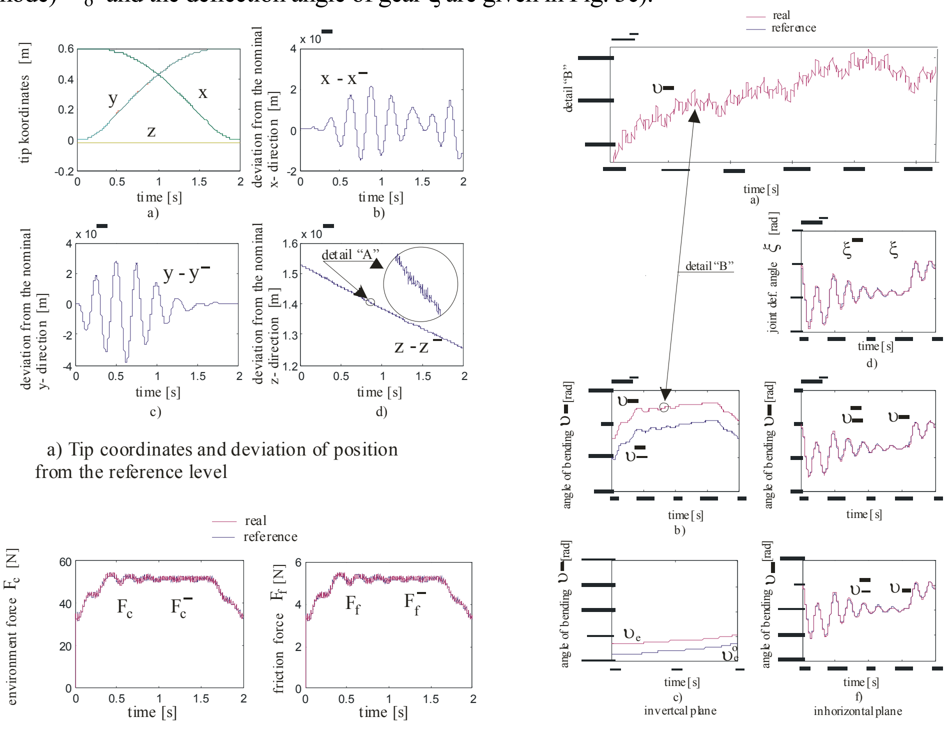
In robotics the reference trajectory is defined in purely kinematics way i.e. geometric and now in the presence of the elasticity elements we can include also the elastic deformation values at the reference level i.e. at the level of knowing the elasticity characteristics during the reference trajectory defining.

The same example analyzed as in paper [9]. Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory. The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link.

$$C_\xi = 0.99 \cdot C_\xi{}^o,\ B_\xi = 0.99 \cdot B_\xi{}^o,\ C_{s1,1} = 0.99 \cdot C_{s1,1}{}^o,\ B_{s1,1} = 0.99 \cdot B_{s1,1}{}^o, C_{s1,2} = 0.99 \cdot C_{s1,2}{}^o,\ B_{s1,2} = B_{s1,2}{}^o / 2.$$

All other characteristics of the system and environment are the same as in paper [9].

As can be seen from Fig. 3a) in its motion from point "A" to point "B" the robot tip tracks well the reference trajectory in the space of Cartesian coordinates. As position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of position from the reference level (see Fig. 3b)). The elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (first mode) ϑ_m and the angle of bending of the upper part of the link (second mode) ϑ_e, as well as elastic deformations taking place in the horizontal plane: the angle of bending of the lower part of the link (first mode) ϑ_q, the angle of bending of the upper part of the link (second mode) ϑ_δ and the deflection angle of gear ξ are given in Fig. 3c).



a) Tip coordinates and deviation of position from the reference level

b) Dynamic force of the environment

c) The elastic deformations

Fig. 3 Simulation results

The rigidity of the second mode is about ten times lower compared with that of the first mode, it is then logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

Fig. 3c) marked a) exhibits the wealth of different amplitudes and circular frequencies of the present modes of elastic elements.

6. CONCLUSIONS

This paper offers a new approach to the analysis of the elastic complex robotic system because it solves the problem of flexibility in a simple way.

In the dynamics of the robotic mechanism all coupling elements are kept, as well as, all dynamic effects of present forces. Elastic deformations of gears and modes are defined as logical consequence of prominent moments of dynamic mechanism of robot as well as dynamics of contact force, which means burdening during the realization of robotic task.

Angles of bent mode, as well as deflection angles of gears, are nonlinear functions of angles of turning drive, like current coordinates of state, and because of their nature cannot be separated as independent coordinate states, neither can dynamic of elasticity be analyzed independently from dynamic of complete robot system under the influence of dynamic environment. We demonstrated that coupling between these magnitudes was strongly presented in complete dynamic movement of the complex robot system. Dynamic model of elastic robotic system was formed by applying Lagrange's equations.

This is just the procedure for obtaining the Euler-Bernoulli equation of flexible line of the first and second mode links. We demonstrated that the equation of motion of all the forces involved at any point (at the link tip on this elastic line also) follows directly from the Euler-Bernoulli equation. If we define boundary conditions for the mode tip as the most interesting point on the elastic line, we obtain the equation of motion at that point, which is the classical form of the mathematical model of the elastic robotic system considered. Thus we demonstrated the connection of the motion equation and the Euler-Bernoulli equation.

The Euler-Bernoulli equation was analyzed from several aspects and one of the very important conclusions is the following:

The elastic deformation is the consequence of the total robot system dynamics which is essentially different from, until now, widely used method that implies the adaptation of the "assumed modes technique".

The Euler-Bernoulli equation was also expanded from several other aspects.

The Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that participate in the formation of the bending moment of the considered mode. Structure of the stiffness matrix must also have the elements outside the diagonal, because of the existence of strong coupling between the elasticity forces involved. Damping is an omnipresent elasticity characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.

With elastic robotic systems, the actuator torque is opposed to the elasticity moment of the first elastic element, which comes after the motor. In our case it is the motor moment that is connected to the rest of the mechanism via the equivalent elasticity moment.

We got this system of differential equations of damped oscillations for observed robotic system. For such defined dynamic model of robotic system we chose the position law of local feedback control.

The choice of the reference trajectory, which depends on the level of the elasticity characteristics knowledge, is analyzed. The estimated elasticity characteristics can be included in the reference trajectory as well as in the control law.

The analogue between the Euler-Bernoulli equation and its solution and modern knowledge in Robotics is presented in this paper.

Based on received simulations it is proved that in this new suggested procedure for analysis of elastic complex robotic systems practically we do not have to worry about how complex a robotic system is and how elastic it is, because both of its characteristics, that are significantly in coupling, are included in "kinematic" as well as in dynamic model. Complete analysis of elastic complex robotic system is done, and that is a unique and completely new approach in solving these problems according to already published and available works.

Acknowledgement: *This research was supported by Ministry of Science and Technological Development of Serbia for financing the national research project "Humanoid robotics: Theory and Application".*

REFERENCES

1. Book, W.J, Maizza-Neto, O, Whitney, D.E. (1975), Feedback Control of Two Beams, Two Joint Systems with Distributed Flexibility, *Trans ASME J. Dyn. Syst. Meas. And Control.* 97 G (4), pp. 424-431.
2. Book, W.J, Majette, M. (1983), Controller Design for Flexible, Distributed Parameter Mechanical Arms via Combined State Space and Frequency Domain Techniques, *Trans ASME Journal of Dynamic Systems, Measurement and Control,* 105, pp. 245-254.
3. Book, W.J. (1984), Recursive Lagrangian Dynamics of Flexible Manipulator Arms, *International Journal of Robotics Research,* Vol. 3. No. 3.
4. Cheong, J, Chung, W, Youm, Y. (April 2000), Bandwidth Modulation of Rigid Subsystem for the Class of Flexible Robots, *Proceedings Conference on Robotics &Automation,* San Francisco, pp. 1478-1483.
5. Cheong, J, Chung, W.K, Youm, Y. (Oct. 2002), PID Composite Controller and Its Tuning for Flexible Link Robots, *Proceedings of the 2002 IEEE/RSJ, International Conference on Intelligent Robots and Systems EPFL, Lausanne, Switzerland*, pp. 2122-2128.
6. De Luca, A, Siciliano, B. (July/August 1991), Closed-Form Dynamic Model of Planar Multilink Lightweight Robots, *IEEE Transactions on Systems, Man, and Cybernetics,* Vol. 21, pp. 826-839.
7. Despotovic, Z, Stojiljkovic, Z. (Feb. 2007), Power Converter Control Circuits for Two-Mass Vibratory Conveying System with Electromagnetic Drive: Simulations and Experimental Results, *IEEE Transactions on Industrial Electronics,* Vol.54, Issue I, pp.453-466.
8. Filipovic, M, Vukobratovic, M. (Nov. 2005), Modeling of Flexible Robotic Systems, *EUROCON 2005 - The International Conference on Computer as a Tool, Belgrade, Serbia and Montenegro.* Vol. 2, pp. 1196-1199.
9. Filipovic, M, Vukobratovic, M. (September 2006), Contribution to modeling of elastic robotic systems, *Engineering & Automation Problems, International Journal,* Vol. 5, No. 1, pp. 22-35.
10. Filipovic, M, Potkonjak, V. Vukobratovic, M. (2007), Humanoid robotic system with and without elasticity elements walking on an immobile/mobile platform, *Journal of Intelligent & Robotic Systems, International Journal,* Vol. 48, pp. 157–186.
11. Filipovic, M, Vukobratovic, M. (June 2008), Complement of Source Equation of Elastic Line, *Journal of Intelligent & Robotic Systems, International Journal,* Volume 52, No. 2, pp. 233 - 261.
12. Filipovic, M, Vukobratovic, M. (Nov. 2008), Expansion of source equation of elastic line, *Robotica, International Cambridge Journal,* Vol. 26, No. 6, pp. 739-751.

13. Hughes, P.C. (1977), Dynamic of a flexible manipulator arm for the space shuttle, *AIAA Astrodynamics Conference, Grand Teton National Park,* Wyoming, USA
14. Jang, H, Krishnan, H, Ang, Jr. M.H. (1997), A simple rest-to-rest control command for a flexible link robot, *IEEE Int. Conf. on Robotics and Automation,* pp. 3312-3317.
15. Khadem, S.E, Pirmohammadi, A.A. (April 2003), Analytical Development of Dynamic Equations of Motion for a Three-Dimensional Flexible Link Manipulator With Revolute and Prismatic Joints, *IEEE Transactions on Systems, Man and Cybernetics, part B: Cybernetics,* Vol. 33, No. 2.
16. Kim, J.S, Siuzuki, K, Konno, A. (April 1996), Force Control of Constrained Flexible Manipulators, *International Conference on Robotics and Automation,* Minneapolis, Minnesota, pp. 635-640.
17. Krishnan, H. (1995), An Approach to Regulation of Contact Force and Position in Flexible-Link Constrained Robots, *IEEE,* Vancouver, pp. 2087-2092.
18. Low, H.K. (June 1987), A Systematic Formulation of Dynamic Equations for Robot Manipulators with Elastic Links, *J Robotic Systems,* Vol. 4, No. 3, pp. 435-456.
19. Low, H.K, Vidyasagar, M. A. (Jun 1988), Lagrangian Formulation of the Dynamic Model for Flexible Manipulator Systems, *ASME J Dynamic Systems, Measurement, and Control,* Vol. 110, No. 2, pp. 175-181.
20. Low, H.K. (Aug 1989), Solution Schemes for the System Equations of Flexible Robots, *J. Robotics Systems,* Vol. 6, No. 4, pp. 383-405.
21. Matsuno, F, Wakashiro, K, Ikeda, M. (May 1994), Force Control of a Flexible Arm, *International Conference on Robotics and Automation, San Diego, CA, USA.* pp.2107-2112.
22. Meirovitch, L. (1967), *Analytical Methods in Vibrations...* Macmillan, New York.
23. Potkonjak, V, Vukobratovic, M. (1999), Dynamics in Contact Tasks in Robotics, Part I General Model of Robot Interacting with Dynamic Environment, *Mechanism and Machine Theory,* Vol. 33, pp. 923-942.
24. Spong, W.M. (1987), Modeling and control of elastic joint robots, *ASME J. of Dynamic Systems, Measurement and Control,* 109, pp. 310-319.
25. Spong, W. M. (Jan. 1989), On the Force Control Problem for Flexible Joint Manipulators, *IEEE Trans. on Autom. Control,* 34(1), pp. 107-111.
26. Surdilovic, D, Vukobratovic, M. (1996), One method for efficient dynamic modeling of flexible manipulators, *Mechanism and Machine Theory,* Vol. 31, No. 3, pp. 297-315.
27. Yim, W. (April 1996), Modified Nonlinear Predictive Control of Elastic Manipulators *International Conference on Robotics and Automation,* Minneapolis, Minnesota, pp. 2097-2102.

DOPRINOS NOVOJ INTERPRETACIJI EULER-BERNOULLI JEDNAČINE

Mirjana Filipović, Vladimir Kokotović

Razmatrano je uvođenje elastičnosti linka u matematički model kompleksnog elastičnog robotskog sistema. koji je u kontaktu sa dinamičkom okolinom. U ovom radu je posvećena je posebna pažnja kinematici i dinamici kretanja elastičnog linka u okviru robotske konfiguracije. Ojler-Bernuli jednačinu kao i njeno rešenje treba proširiti prema zahtevima složenosti kretanja robotskog sistema. Elastična deformacija kako po amplitude tako i po frekvenciji je dinamička veličina koja zavisi od ukupne dinamike kretanja robotskog sistema. Matrica krutosti (i prigušenja) je puna matrica. Matematički model aktuatora takođe obuhvata kuplovanje između sila elastičnosti. Superponiranjem partikularnog rešenja oscilatornog karaktera (integral Danijela Bernulija) i stacionarnog rešenja prinudnog karaktera, mi možemo da dobijemo bilo koju elastičnu deformaciju posmatranog moda. Opšta forma elastične linije direktno sledi iz dinamike kretanja sistema.

Ključne reči: *modeliranje, robot, elastična deformacija, sprezanje, kinematka, dinamika.*