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PROBABILITY OF STABILITY AND RELIABILITY OF DISCRETE DYNAMICAL SYSTEMS

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Abstract. *Method for the probability of stability estimation of discrete systems with randomly chosen parameters is presented in this paper. Various formulas for different types of parameter distribution are obtained and used to calculate probability of stability for an arbitrary order system. Since the main goal is to obtain the system maximum probability of stability, using this method and choosing the adequate values of parameters it is possible to do so. For its simplicity and efficiency this method can be used in practical applications and it is appropriate for reliability analysis of discrete systems with variable parameters.*

Key words: *Probability of stability, reliability, imperfect system*

1. INTRODUCTION

Systems with random parameters exist in many industries, for example: process, chemical, rubber, plastic materials industry, etc. Often these systems have some difficulties in working properly since their parameters values, usually, differ from the wanted ones. In this case estimating the influence of these stochastic parameters on the system performances in advance is of great importance. This estimation is necessary for the analysis of system stability, reliability, quality of the system, etc.

These systems are called imperfect systems. Stability problem of systems with imperfections is well known. In this paper stability problem of linear discrete systems with parametric imperfection is considered.

Stability of the system is determined by value of system parameters. If parameters have constant values, system can be stable or unstable. If parameters are stochastic, system is stable with some probability called probability of stability.

Stochastic stability plays an important role in researching the behaviour of dynamic and control systems under random parametric variations. Several modes of stochastic stability are known: stability of probability, stability of the K – th moment, almost sure sta-

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bility, Lyapunov average stability, monotonic entropy stability, etc. For all definitions of stochastic stability it is necessary for probability of stability to be equal to 1.

Caughey and Gray, [1], determinated almost sure stability of linear dynamic systems with stochastic coefficients. Khasminski, [2], Kozin, [3, 4], and Pinsky, [5], gave different definitions and characteristics of stochastic stability of ordinary differential equations. Porter and Crossley, [6], analyzed the probability of stability of a class of linear dynamical systems and gave some theoretical results. Stochastic stability of systems with random imperfections is analyzed in [7].

The basic methods for the probability of stability estimation of continuous systems are given in [8-11]. In [12] the method for the probability of stability estimation of discrete systems with random parameters is presented. Some theorems from the theory of the random processes and the basic conditions for the discrete systems stability are used in [13] [14], respectively. In this paper we considered randomly chosen and time invariable parameters.

The probability of stability estimation can be used for reliability and robustness estimation, as well.

The probability of stability is in correlation with reliability. If parameters have stationary values in time interval t, then in that time interval the system is reliable. The value of parameter can be changed under the influence of different factors, exterior and interior, i.e., wear, corrosion, aging, etc. The effects of these changes depend on components quality, [15], and can decrease the system reliability causing the failure of the component and system as well.

Robustness estimation can be accomplished using the probability of stability estimation, [16]. In the case of constant parameters, the results obtained using the method for probability of stability estimation, are equivalent to the results obtained by Kharitonov's method, [17]. Method for probability of stability estimation represents the generalization of Kharitonov's method when the system parameters are uniformly distributed random variables.

The probability of stability estimation of linear discrete systems with random parameters is given in [13], [18], [19]. In [20] the failure probability of the system has been considered. The presented method, [12], can be applied for the analysis of the continuous systems too [21].

The great importance of this method is its application in practice. The selection of adequate values of parameters provides the maximum probability of stability. This way many problems can be avoided, for example failures of components and system. This method can be applied for different parameters distributions such as uniform, normal, exponential, Poisson distribution.

2. Probability of Stability Estimation

The mathematical model of the nth order discrete system is given by:

$$\sum_{i=0}^{n} l_i \quad x(kT+n-i)=0, \;\; T=1, \;\; l_0=1 \tag{1}$$

where parameters l_i are random variables with probability distribution densities $p_i(l_i)$ and T is the sampling time.

First we have to determine stability region of the system (1) in the parametric space. The characteristic polynomial of the difference equation (1) is:

$$z^n + l_1 z^{n-1} + \cdots + l_n = 0 \tag{2}$$

The necessary and sufficient condition for the stability of system (1) is that all zeros of its characteristic polynomial are located inside the unit circle in the z – plane. To test this condition the bilinear transform method is used mapping the inside of the unit circle into the left half of the complex plane. The new equation is obtained with coefficients $\varphi_i(l_1, l_2, \ldots, l_n) = \varphi_i$:

$$s^n + \varphi_{n-1} s^{n-1} + \cdots + \varphi_0 = 0 \tag{3}$$

Hurwitz criterion is applied to obtain stability region S_n. The necessary and sufficient condition that all zeroes of equation (3) are located in the left half of the complex plane is that all diagonal minors D_i, of Hurwitz matrix D are greater than zero so the stability region S_n is determinated from the next relations $D_1 = \varphi_{n-1} > 0$; $D_2 = \begin{vmatrix} \varphi_{n-1} & \varphi_{n-3} \\ \varphi_n & \varphi_{n-2} \end{vmatrix} > 0$, etc.

For the second order system the stability region, S_2, is given by the next set of inequalities:

$$\begin{aligned} &1 - l_1 + l_2 \geq 0 \\ &1 + l_1 + l_2 \geq 0 \\ &l_2 \leq 1 \end{aligned} \tag{4}$$

The stability region S_2 is presented in Fig. 1, where N_2 presents the unstable region. In the case of the second order system, the stability region is a triangle.

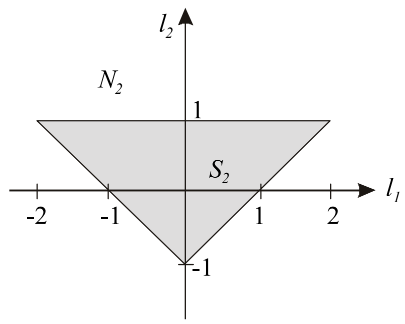

Fig. 1 Stability region, S_2, of the second order discrete system

For the third order system, the stability region, S_3, is given by the next set of inequalities:

$$\begin{aligned} &l_1 + l_2 + l_3 > -1 \\ &l_1 - l_2 + l_3 < 1 \\ &l_1 l_3 + 1 > l_2 + l_3^2 \end{aligned} \tag{5}$$

The stability region is shown in Fig. 2.

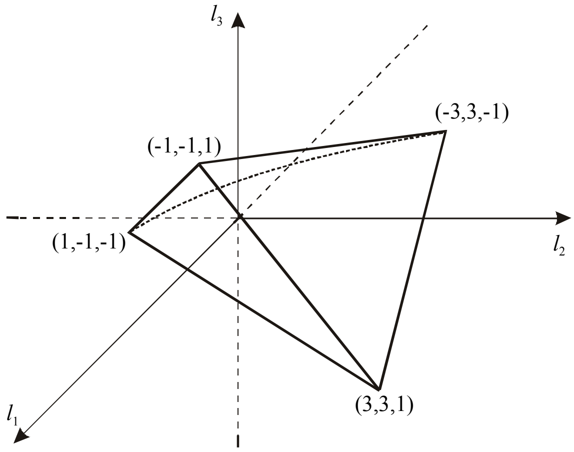

Fig. 2 Stability region, S_3, of the third order discrete system

The total distribution density is given by:

$$p(l_1,\ldots,l_n)=\prod_{i=1}^{n} p_i(l_i) \tag{6}$$

and probability of stability of system (1) is:

$$P=\int\underset{S_n}{\cdots}\int p(l_1,\ldots,l_n)\quad dl_1\cdots dl_n \tag{7}$$

In the case when stability region is determined for more parameters problem becomes complex. Hence, surfaces limiting the stability region are usually defined by very complex mathematical relations. It is, also, necessary to integrate by the area S_n, (7), which is very complicated considering the complex distribution densities of random parameters.

In the case of the n^{th} order system, the stability region is determined by the known conditions for the stability of the n^{th} order discrete systems. This way enclosed body is obtained, presenting the stability region in the n^{th} order parametric plane. This body is very complex and the calculation of the probability of stability, (7), is very difficult. So it is important to estimate the probability of stability for the practical applications. For higher order system this estimation is very rough, but significant in practice and it is the best possible estimation so far.

For the probability of stability estimation next theorems can be applied effectively.

Theorem 1. The stability region, S_n, of difference equation (1) belongs to the region $\overline{P}_n$ (hyper parallelepiped) given by:

$$|l_i|\le\binom{n}{i},\ i=1,2,\ldots,n \tag{8}$$

The stability region is limited above by the region $\overline{P}_n$ in the parametric plane, $S_n\in\overline{P}_n$.

Theorem 2. The stability region, S_n, of difference equation (1) comprises the region $\underline{P}_n$ (simplex polyhedron) given by:

$$|l_1|+|l_2|+\ldots+|l_n|\le 1 \tag{9}$$

The stability region S_n is limited lower by the region $\underline{P}_n$, $\underline{P}_n \in S_n$.

This means that the stability region S_n is inside the region $\overline{P}_n$, and the region $\underline{P}_n$ is inside the stability region S_n. The proofs of theorems 1 and 2 are given in [12].

Using these theorems, the probability of stability of n[th] order system can be estimated in the following way:

$$\iint_{\underline{P}_n}\cdots\int p(l_1,\cdots,l_n)dl_1\cdots dl_n < P < \iint_{\overline{P}_n}\cdots\int p(l_1,\cdots,l_n)dl_1\cdots dl_n \tag{10}$$

i.e.

$$P_{\underline{P}_n} < P < P_{\overline{P}_n} \tag{11}$$

$P_{\underline{P}n}$ is the probability that the stability region lies inside the region $\underline{P}_n$ and $P_{\overline{P}_n}$ is the probability that the stability region lies inside the region $\overline{P}_n$.

Since it is difficult to estimate probability of stability because of the complexity of region $\underline{P}_n$ it is better to choose constant limits of region $\underline{P}_n$, which is supported by the next theorem.

Theorem 3. The n^{th} order polynomial $f(z)=z^n+l_{n-1}z^{n-1}+\cdots+l_1z+l_0$ is given. All zeroes of the polynomial are in the circle of radius $R=\max\left[\frac{|l_{n-k}|}{|\lambda_k|}\right]^{1/k}$, where λ_k are real numbers satisfying the condition $\sum_{k=1}^{n}|\lambda_k|=1$, [22].

Since the stability region of discrete systems is the unit circle, using theorem 3, for $R = 1$ the next relations are obtained:

$$1=\max\left[\frac{|l_{n-k}|}{|\lambda_k|}\right]^{1/k} \tag{12}$$

$$\left[\frac{|l_{n-k}|}{|\lambda_k|}\right]^{1/k}\le 1 \qquad k=1,\cdots,n \tag{13}$$

$$\left[\frac{|l_{n-k}|}{|\lambda_k|}\right]\le 1 \tag{14}$$

$$|l_{n-k}|\le|\lambda_k| \tag{15}$$

Summing the following relation is obtained:

$$\sum_{k=1}^{n}\left|l_{n-k}\right| \le \sum_{k=1}^{n}\left|\lambda_k\right| = 1 \tag{16}$$

This result is equal to the relation (9). Using theorem 3, constant limits for region $\underline{P}_n$ is chosen, i.e. $\lambda_k = 1/n$. In this case the region $\underline{P}_n$ is hyper parallelepiped defined as $|l_{n-k}| \le 1/n$, where n is order of the system.
In previous work, [12], using relation (9), formula for calculating probability $P_{\underline{P}n}$ for the n^{th} order discrete system, could have been obtained only for the uniform distribution of parameters. For other probability distributions, probability $P_{\underline{P}n}$ could have been calculated only for the second and third order discrete systems. For higher order systems calculation of $P_{\underline{P}n}$ was very complicated because of the complexity of the region $\underline{P}_n$. Now, using theorem 3 and choosing the constant limits of region $\underline{P}_n$ it is possible to obtained formulas for calculating $P_{\underline{P}n}$ for other parameters distributions besides uniform. This is the novelty in relation to the results obtained in [12] and the great contribution in the paper.
In the case of the uniform distribution of the parameters:

$$p_i = \begin{cases} \dfrac{1}{(\overline{a}_i - \underline{a}_i)}; & \underline{a}_i < a < \overline{a}_i \\ 0\ ; & a > \overline{a}_i;\ \ a < \underline{a}_i \end{cases} \tag{17}$$

The next probabilities are obtained for the n$^{\text{th}}$ order discrete system:

$$P_{\overline{P}_n} = \frac{\prod_{i=1}^{n}\left\{\left[\left|\overline{a}_i + \binom{n}{i}\right| - \left|\overline{a}_i - \binom{n}{i}\right|\right] - \left[\left|\underline{a}_i + \binom{n}{i}\right| - \left|\underline{a}_i - \binom{n}{i}\right|\right]\right\}}{2^n \prod_{i=1}^{n}\left(\overline{a}_i - \underline{a}_i\right)} \tag{18}$$

$$P_{\underline{P}_n} = 1 - \frac{1}{n}\cdot\frac{\sum_{i=0}^{2^n-1} f_{ij}^{n}\cdot h(f_{ij})}{\prod_{i=1}^{n}(\overline{a}_i - \underline{a}_i)} \tag{19}$$

where $\overline{a}_i$ and $\underline{a}_i$ present the upper and lower limit of the interval, and f_{ij} is obtained using the formula:

$$f_{ij} = \sum_{j=1}^{n}\left\{\left(\frac{1+(-1)^{\left[\frac{i}{2^{j-1}}\right]}}{2}\right)\cdot|\,\overline{a}_j\,| + \left(\frac{1-(-1)^{\left[\frac{i}{2^{j-1}}\right]}}{2}\right)\cdot|\,\underline{a}_j\,|\right\} - 1 \tag{20}$$

This formula is proper for $f_{ij} \le 1$. For other cases, probability $P_{\underline{P}n}$ is obtained from the intersection of the region $\underline{P}_n$ and the uniform distribution (graphical presentation of the

uniform distribution for the n^{th} order system is a hyper parallelepiped) which is very complicated because of the shape of the intersections.
For the n^{th} order system whose parameters are random, normally distributed variables, with $p_i = \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(l_i - l_{mi})^2}{2\sigma_i^2}}$, probabilities $\overline{P}_n$ and $\underline{P}_n$ can be calculated from the following equations:

$$P_{\overline{P}_n} = \left(-\frac{1}{2}\right)^n \prod_{i=1}^{n} \left[\Phi\left[\frac{-\binom{n}{i} - l_{mi}}{\sqrt{2}\sigma_i} \right] - \Phi\left[\frac{\binom{n}{i} - l_{mi}}{\sqrt{2}\sigma_i} \right] \right] \tag{21}$$

$$P_{\underline{P}_n} = \left(-\frac{1}{2}\right)^n \prod_{i=1}^{n} \left[\Phi\left[\frac{-\frac{1}{n} - l_{mi}}{\sqrt{2}\sigma_i} \right] - \Phi\left[\frac{\frac{1}{n} - l_{mi}}{\sqrt{2}\sigma_i} \right] \right] \tag{22}$$

where Φ is the Laplace function.
For exponential distribution of parameters probability of stability can be estimated using relations:

$$P_{\overline{P}_n} = \left(e^{-\sum_{i=1}^{n-1} \binom{n}{i+1} \cdot l_{i+1}} - e^{-\sum_{i=1}^{n} \binom{n}{i} \cdot l_i} \right) \cdot \prod_{i=1}^{n-1} \left(-1 + e^{\binom{n}{i+1} \cdot l_{i+1}} \right) \tag{23}$$

$$P_{\underline{P}_n} = \left(e^{-\left(\frac{1}{n}\right) \cdot \sum_{i=1}^{n-1} l_{i+1}} - e^{-\left(\frac{1}{n}\right) \cdot \sum_{i=1}^{n} l_i} \right) \cdot \prod_{i=1}^{n-1} \left(-1 + e^{\frac{1}{n} \cdot l_{i+1}} \right) \tag{24}$$

For Poisson distribution of parameters, the next relations are obtained:

$$P_{\overline{P}_n} = \prod_{i=1}^{n} \frac{1}{k_i!} \left(\Gamma(1+k_i) - \Gamma\left(1+k_i \; ; \; \binom{n}{i} l_i \right) \right) \tag{25}$$

$$P_{\underline{P}_n} = \prod_{i=1}^{n} \frac{1}{k_i!} \left(\Gamma(1+k_i) - \Gamma\left(1+k_i \; ; \; \frac{1}{n} \cdot l_i \right) \right) \tag{26}$$

where $\Gamma(1+k_i)$ is gamma function, and $\Gamma\left(1+k_i \, ; \binom{n}{i} l_i\right)$ (or $\Gamma\left(1+k_i \, ; \frac{1}{n} \cdot l_i\right)$) incomplete gamma function.
For the first, second and third order discrete system probability of stability can be calculated exactly, but for the higher order discrete systems using these formulas the probability of stability can be estimated.

3. The Probability of Stability Application for Reliability Analysis

The method for probability of stability estimation can be used for reliability analysis. There are many reasons for reliability decrease and in this paper we consider the case when system instability is the cause of it. It is assumed that the system is reliable as long as it is stable.

For the higher order systems stability region is very complex and it is limited from above by hyper parallelepiped, and from below by simplex polyhedron. A system is reliable if it is situated in the stability region, so it is necessary to determinate the conditions that will enable the system to remain stable during certain time interval. In order to find adequate probability density function of random variable, which is in our case the value of system parameter, we have used the example of onedimensional Brownian motion of particles. It can be proved that random variable has approximatelly $N(0;\sigma^2 t)$ probability distribution, [23]. For the second and the third order discrete systems it is possible to calculate reliability exactly, with results given in the next figures.

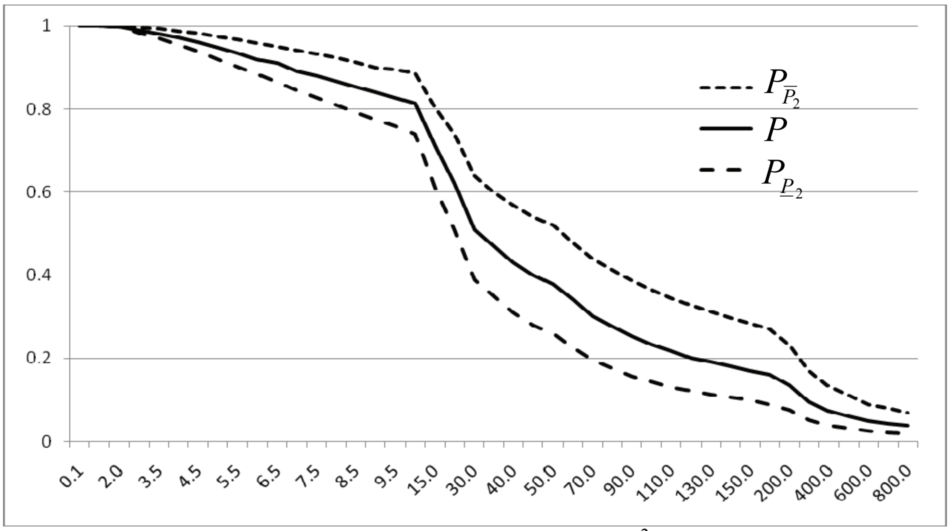

Fig. 3 Results for the second order discrete system with $N(0;\sigma^2 t)$ distribution of parameters

Values $P_{\underline{P}_n}$, P and $P_{\overline{P}_n}$, where P is the probability of system work without failures, i.e. system reliability, correspond to relation (11). For the higher order discrete systems it is difficult to calculate reliability because of complexity of stability region. In that case, reliability must be estimated using the method for the probability of stability estimation. This estimation has significant value in practical applications.

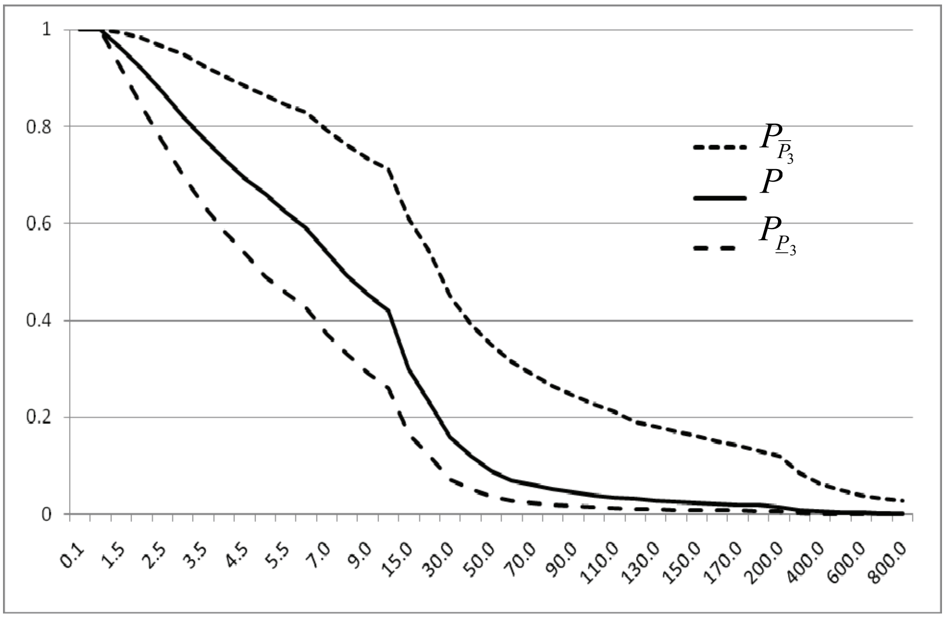

Fig. 4 Results for the third order discrete system with $N(0;\sigma^2 t)$ distribution of parameters

4. Conclusion

The method to perform the probability of stability estimation of discrete systems with randomly chosen parameters is described. For different types of parameter distributions, different formulas for the probability of stability estimation are obtained. Using these formulas the probability of stability of the arbitrary order systems can be calculated. This paper considers real – time technique, based on the probability of stability, for reliability analysis of discrete systems with parameter imperfections. Results are illustrated with the example of the second and third order discrete systems.

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VEROVATNOĆA STABILNOSTI I POUZDANOST DISKRETNIH DINAMIČKIH SISTEMA

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U ovom radu je predstavljen metod za procenu verovatnoće stabilnosti diskretnih sistema sa slučajno izabranim parametrima. Izvedene su formule za izračunavanje verovatnoće stabilnosti sistema proizvoljnog reda za različite raspodele parametara. Pošto je glavni cilj dobiti maksimalnu verovatnoću stabilnosti, korišćenjem ovog metoda i izborom adekvatnih vrednosti parametara to je moguće ostvariti. Zbog svoje jednostavnosti i efikasnosti ovaj metod se može koristiti i u praktičnim primenama i pogodan je za analizu pouzdanosti diskretnih sistema sa promenljivim parametrima.

Ključne reči: *Verovatnoća stabilnosti, pouzdanost, nesavršeni sistemi.*