FACTA UNIVERSITATIS
Series: **Automatic Control and Robotics** Vol. 8, N° 1, 2009, pp. 35 - 44

ON A JERK DYNAMICAL SYSTEM

UDC 517.9 517.93 681.5

Ljubiša M. Kocić[1], Sonja Gegovska-Zajkova[2]

[1]Faculty of Electronic Engineering, University of Nis, Nis, Serbia
E-mail: ljubisa.kocic@elfak.ni.ac.rs
[2]Faculty of Electrical Engineering and Information Technologies, University Ss Cyril and Methodius, Skopje, Macedonia, E-mail: szajkova@feit.ukim.edu.mk

Abstract. *Chaotic systems of J.C. Sprott* [7-11] *emanated from electric circuits turn to be attractive examples of week chaos - the only form of chaos that eventually might be acceptable in sensible applications like automatic control or robotics. Here, two modifications of a 3D dynamic flow, known as jerk dynamical system of J.C. Sprott* [9], *are considered.*

Key words: *jerk dynamics, chaos, differential equation*

1. INTRODUCTION

One of the problems that lead to a better understanding of chaotic phenomena is to find the simplest possible mathematical model of a dynamical system that still may have chaotic behaviour. If applied to continuous flows of the type $d\mathbf{u}/dt = f(\mathbf{u})$, where **u** is an *n*-dimensional vector containing time dependent functions and *f* is a smooth function, the famous Poincaré-Bendixon theorem [1] claims that such a system can have bounded chaotic solution only for $n \geq 3$. In 1963, Lorenz [2] showed that chaos can occur in 3D systems of autonomous ordinary differential equations with seven terms and two quadratic nonlinearities:

$$\begin{aligned}\dot{x} &= 10(y-x),\\ \dot{y} &= (28-z)x-y,\\ \dot{z} &= xy-8z/3.\end{aligned}$$

Later, in 1976, Rössler [3] simplified Lorenz stetting by reducing quadratic nonlinearities to one, proposing the following chaotic system:

Received April 23, 2010

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x + 0.2y,\\ \dot{z} &= 0.2 + xz - 5.7z.\end{aligned}$$

Three years later [4] he restricted the number of terms to six, obtaining a system with a single quadratic nonlinearity and two parameters:

$$\begin{aligned}\dot{x} &= -y - z,\\ \dot{y} &= x,\\ \dot{z} &= 0.386y(1-y) - 0.2z,\end{aligned}$$

which exhibits chaotic dynamics as well. For some other values of the parameters, the dynamics becomes quasiperiodic, with trajectory that lies on an invariant torus.

Trying to simplify Rössler's model, Sprott [7] founded fourteen algebraically distinct cases with six terms and a single nonlinearity (cases F - S, p. R648), and five cases with five terms and two nonlinearities. One of the consequences of simplicity is that the majority of these systems can be written in the form of an explicit 3rd-order ordinary differential equation describing the time evolution of a single scalar variable x according to:

$$\dddot{x} = J(\ddot{x},\ \dot{x},\ x). \tag{1}$$

Such systems have been called "jerk systems" because the third order time derivative of the displacement in mechanical systems is called *jerk*. Thus, in order to study different aspects of chaos, differential equation (1) can be considered instead of a 3D system. Note that reduction of the dynamical system to a jerk form for each of the phase variables is not always an easy task. Sprott's work inspired Gottlieb [5] to pose the question of finding the simplest jerk function that generates chaos.

In 1997 Linz [6] showed that both Lorenz and Rössler models can be written down in jerk form, although a bit complicated. The same year, Sprott [8] proposed chaotic jerk circuit containing just 5 terms and one quadratic nonlinearity (A is parameter):

$$J(\ddot{x},\ \dot{x},\ x) = -A\ddot{x} + \dot{x}^2 - x, \tag{2}$$

which seems to be the simplest quadratic jerk function that is able to produce chaos for $2.0168 < A < 2.0577$. The system has the greatest Lyapunov exponent $\lambda \cong 0.0551$, for $A = 2.017$, which implies maximal chaos in this "minimal chaotic system" ([9], [12]).

Trying to simplify (2), Sprott experimented with replacing linear term x in J with several piecewise linear functions, considering jerk functions of the form

$$J(\ddot{x},\ \dot{x},\ x) = A\ddot{x} + B\dot{x} + \varphi(x), \tag{3}$$

where A and B are real parameters and $\varphi(x)$ is a simple nonlinear function. The results were published in the series of papers (see [8], [10], [11], [12]), but equation (2) remains the simplest jerk equation with quadratic nonlinearity.

Note that system (1) describes an autonomous dissipative dynamic flow that can be interpreted also as a damped harmonic oscillator driven by a nonlinear external energy. Indeed, by combining (1) and (3), one gets $\dddot{x} + A\ddot{x} + B\dot{x} = \varphi(x)$, which, after integrating yields $\ddot{x} + A\dot{x} + Bx = \int \varphi(x)\,dt$. This explains chaotic behaviour of such systems.

In [12] Sprott and Linz have experimented with reducing the quadratic term $\dot{x}^2$ in (2) by $|\dot{x}|$, ending up with a modified jerk function

$$J(\ddot{x},\ \dot{x},\ x) = -A\ddot{x} + |\dot{x}| - x\,. \tag{4}$$

But, as it is shown in [12] this replacement leads to a non-chaotic system. Then, they have searched for pairs (a, b) such that $\dddot{x} = -a\ddot{x} + |\dot{x}|^b - x$ would have a chaotic solution. For other experiments with Sprott type chaotic systems see [13]-[18].

The intention of present note is to investigate existence of chaotic trajectories in a jerk dynamical system that is placed somehow in between those with jerk functions (2) and (4). In other words, the object of investigation here will be the next two jerk systems

$$\dddot{x} = -A\ddot{x} + g_k(\dot{x}) - x\,, \tag{5}$$

and

$$\dddot{x} = -A\ddot{x} + h_k(\dot{x}) - x\,. \tag{6}$$

where the functions g_k and h_k are defined as a compromise between nonlinear terms $\dot{x}^2$ and $|\dot{x}|$, namely

$$g_k(\xi) = \begin{cases} \xi^2, & \xi \le 0, \\ k\xi, & \xi > 0, \end{cases} \quad \text{or} \quad h_k(\xi) = \begin{cases} -k\xi, & \xi \le 0, \\ \xi^2, & \xi > 0, \end{cases} \tag{7}$$

where $k \ge 0$ is a real parameter. The function g_k can be called *left semi-quadratic* and h_k - *right semi-quadratic*. Related graphs are given in Fig. 1.

2. METHOD

Note that both systems (5) and (6) depend on two parameters $A \in \Re$ and $k \in \Re^+$. The functions g_k and h_k are continuous on all $\Re$, having semi-parabolas as their graphs (Fig. 1). Thus, dynamical systems (5) and (6) fulfill conditions of the Poincaré-Bendixon theorem which means possibility of the existence of chaotic solutions.

The method of verifying existence of chaos includes *four* actions: 1° Construction of trajectories of a given system in phase 3D space $(\dot{x},\ \ddot{x},\ \dddot{x})$; 2° Visualization of at least one of the variable's solution, say $x(t)$, for $0 \le t \le t_{max}$; 3° Finding the discrete Fourier power spectra of $x(t)$; 4° Evaluation of the first Lyapunov coefficient λ_1, which in the case of chaos should be strictly greater than zero.

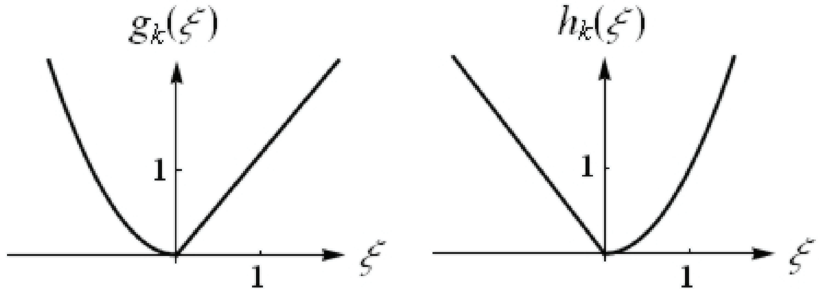

Fig. 1 Left semi-parabola and right semi-parabola

Trajectories in phase space $(\dot{x}, \ddot{x}, \dddot{x})$. After recasting the jerky equations (5) and (6) into the first order ODJ in $\Re^3$, (5) and (6) respectively becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -Az + g_k(y) - x, \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= -Az + h_k(y) - x. \end{aligned} \tag{9}$$

It needs to calculate trajectory $\{\mathbf{u}(t) = (x(t),\ y(t),\ z(t)),\ 0 \le t \le t_{\max}\}$, $\mathbf{u} \in \Re^3$, as a solution of the vector-valued ODJ $\dot{\mathbf{u}} = f(t,\ \mathbf{u}),\ \mathbf{u}(0) = \mathbf{u}_0$ starting from the initial conditions $\mathbf{u}(0) = (x(0),\ y(0),\ z(0))$. For this purpose, after fixing $t_{\max} > 0$, the numeric `NDSolve` subroutine in Wolfram's *Mathematica* package is used, with adaptive embedded pairs of explicit Runge-Kutta method of fourth order that generates the sequences

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{1}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4);\quad t_{n+1} = t_n + s$$

where $\mathbf{u}_{n+1}$ is the Runge-Kutta approximation of $\mathbf{u}(t_{n+1})$, and $s > 0$ is the step. The vector-valued coefficients $\mathbf{k}_i \in \Re$are evaluated as

$$\mathbf{k}_1 = f(t_n,\ \mathbf{u}_n), \qquad \mathbf{k}_2 = f\left(t_n + \frac{1}{2}h,\ \mathbf{u}_n + \frac{1}{2}h\,\mathbf{k}_1\right),$$

$$\mathbf{k}_3 = f\left(t_n + \frac{1}{2}h,\ \mathbf{u}_n + \frac{1}{2}h\,\mathbf{k}_2\right),\quad \mathbf{k}_4 = f(t_n + h,\ \mathbf{u}_n + h\,\mathbf{k}_3).$$

The maximal number of steps is not limited. The fourth-order of the method, means that the error per step is $O(s^5)$, while the total accumulated error has order s^4. The output of the `NDSolve` is interpolating function for each variable, interpolating the discrete set $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{u}_2$, …. To construct the trajectory, it is enough to employ `ParametricPlot3D` which is the built-in function.

Visualization of $x(t)$. Since the time interval is defined as $[0, t_{max}]$ and the components of $\mathbf{u}(t)$ are available as smooth interpolating functions, the rest of the action accomplishes in using `Plot` built-in function in the interval $[0, t_{max}]$.

Discrete Fourier spectra of $x(t)$. Before computing the DFT transform (in *Mathematica* it is doing by using the `Fourier the` built-in function) `the` function $x(t)$ is to be discretized to get the sequence $\{x_1, x_2,...x_n\}$. By calling `Fourier`, the sequence $\{x_1, x_2,...x_n\}$ maps into $\left\{\frac{1}{\sqrt{n}}\sum_{j=1}^{n} x_j \exp\left(\frac{2\pi i}{n}(j-1)(\nu-1)\right),\ \nu = 1,...,n\right\}$. After getting this discrete Fourier transform, the function $|\cdot|^2$ yields the power spectra. But, for the sake of speeding up the procedure, the squaring operation can be omitted without losing the point.

Lyapunov exponent. Finally, the first (greatest) Lyapunov exponent λ_1, is the most significant indicator of chaos, although the most difficult to calculate. According to [8], the system (2) for $A = 2.017$, has $\lambda_1 = 0.0550$ as the first Lyapunov exponent. Theoretically, if $\lambda_1 \geq 0$ the system is chaotic, since two nearby trajectories diverge by the average rate $\exp(\lambda_1)$. However, Sprott and Linz have established the following numerical criteria for chaotic behavior in [12]. They wrote: "chaos is assumed to exist if the largest Lyapunov exponent exceeds 0.005 after 4×10^5 fourth-order Runge-Kutta iterations with a step size of 0.05."

The evaluation of the Lyapunov exponent usually has two stages. The first stage is searching for the *exact* (*fiducial*) *trajectory*. Namely, the starting point $\mathbf{R}_0$ for integrating the system (1) may be any point in the vicinity of the trajectory. The usual choice is $\mathbf{X}_0 = (0, 0, 1)$. After 104 iterations by the fourth order Runge-Kutta with the step 0.002, point $\mathbf{R}_0 = (x(0), y(0), z(0))$ is obtained. One can assume that this point is very close to the fiducial trajectory, and point $\mathbf{R}_0$ can be considered as a fiducial point. This means that R_0 is a good initial point for further calculation of the Lyapunov exponent.

The second stage of Lyapunov exponent evaluation uses the fiducial point $\mathbf{R}_0$ and a nearby point $\mathbf{S}_0$ obtained by perturbation of $\mathbf{R}_0$ by a small amount. Here, the choice $\mathbf{S}_0 = \mathbf{R}_0 + (\varepsilon, 0, 0)$, is used with $\varepsilon = 10^{-10}$. Then, by the same fourth order Runge-Kutta scheme, the new pair of points $(\mathbf{R}_1, \mathbf{S}_1)$ is obtained. Calculate $\lambda = \ln(|\mathbf{R}_1 - \mathbf{S}_1| / |\mathbf{R}_0 - \mathbf{S}_0|)$, set $\mathbf{R}_0 := \mathbf{R}_1$, $\mathbf{S}_0 := \mathbf{R}_1 + \varepsilon \mathbf{S}_1 - \mathbf{R}_1 / \mathbf{S}_1 - \mathbf{R}_1 |$, and start the new round of calculation using $(\mathbf{R}_0, \mathbf{S}_0)$ as the new starting pair. The new λ is obtained as $\lambda := \lambda + \ln(|\mathbf{R}_1 - \mathbf{S}_1| / |\mathbf{R}_0 - \mathbf{S}_0|)$.

After N_{max} iterations (here $N_{max} = 500$) the Lyapunov exponent (in base e) is the arithmetic mean

$$\lambda_1 = \lambda / N_{max}.$$

For testing this algorithm, the biggest Lyapunov exponent for the system described by equation (2) is calculated. The value obtained was exactly the same as is given in [8], namely $\lambda_1 = 0.0550$.

Now, the above four criteria will be applied to trace if the semi-quadratic systems may permit chaotic behavior.

3. NUMERICAL RESULTS

The results are somehow unexpected. Namely, the system (5) (or (8)) with the left-semi quadratic function $g_k(\xi)$, given by (7), undergoes chaotic dynamic for some values of parameters A and k, while the system (6) (or (9)) governed by right-semi quadratic function $h_k(\xi)$ exhibits no chaos at all.

The left-semi quadratic system $\dddot{x} = -A\ddot{x} + g_k(\dot{x}) - x$. The system preserves chaotic regime of the original Sprott setting $\dddot{x} = -A\ddot{x} + \dot{x}^2 - x$ for lesser values of A and bigger slope values k. The choice A = 1.3 and k = 1 produces recognizable phase diagrams, given in two projections in Fig. 1. The $x(t)$ diagram and the DFT confirms chaotic element, and the Lyapunov coefficient is $\lambda_1 = 0.0338091$.

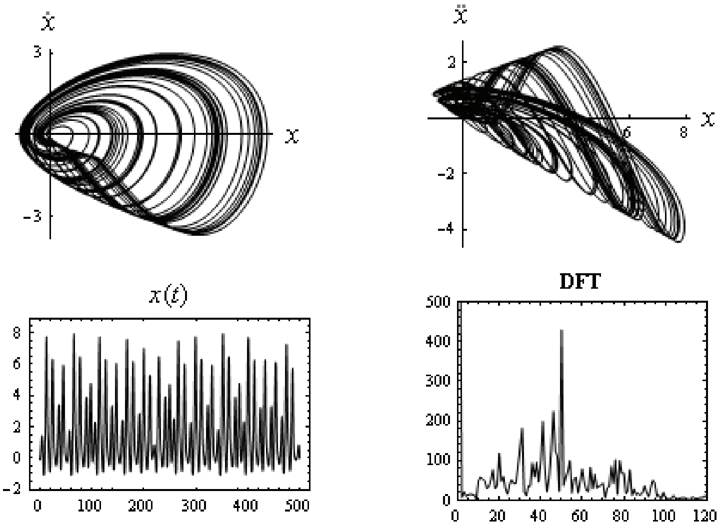

Fig. 2 A = 1.3; k = 1; $\lambda_1 \cong 0.0338091$ (chaos)

The relations between parameters A and k in chaotic regime are given in Table 1.

Table 1. Lyapunov λ_1 exponent for dynamical system (5)

A	k	λ_1
0.9	1.5	0.0554254
1.1	1.25	0.0386954
1.3	1.0	0.0338091
1.3	0.85	0.0421858
1.3	0.75	0.0286423

Lowering the value of slope of the linear part of $g_k(\xi)$ leads towards an aperiodic dynamics, as it is illustrated in Fig. 3.

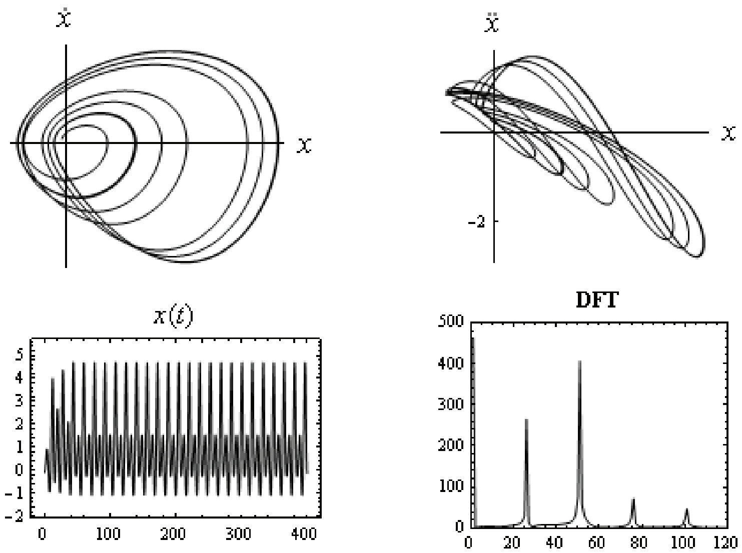

Fig. 3 A =1.3; k = 0.6; λ_1 = 0.0011777 (aperiodic regime)

On the other hand, keeping the slope fixed, and increasing A makes the system get out of chaotic regime, as it is shown in Table 2.

For k = 0.75 the range of A when the system maintain to be chaotic narrows, and centers around the value A = 1.3. Further decreasing of k leads to the periodic regime, which has characteristic limit circles. The Lyapunov exponents are given in Table 3.

Table 2. Lyapunov λ_1 exponent for dynamical system (5) with fixed slope

A	k	λ_1	
1.32	0.75	0.0260195	chaos
1.34	0.75	0.0173497	
1.36	0.75	0.00746725	regular flow
1.38	0.75	–0.00335217	
1.40	0.75	–0.00309719	

Table 3. Lyapunov λ_1 exponent for dynamical system (5) with fixed A

A	k	λ_1
1.3	0.6	0.0011777
1.3	0.35	0.00189326
1.3	0.3	0.00157308
1.3	0.2	–0.00036432
1.3	0.1	0.000204812
1.3	0.0	0.00125791

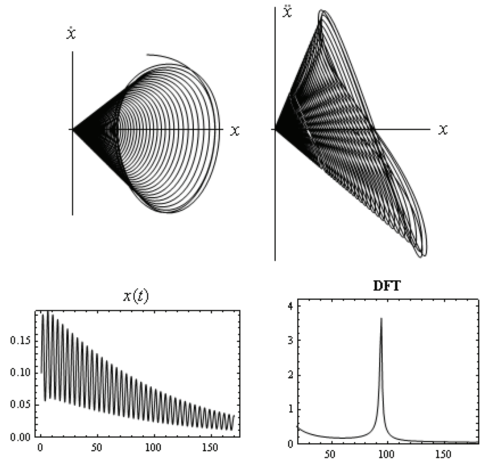

Fig. 4 A = 0.47; k = 4 (damped oscillations)

The right-semi quadratic system $\dddot{x} = -A\ddot{x} + h_k(\dot{x}) - x$. For this system no chaotic regime has been found. For all the chosen combinations of pairs (A, k), system persisted in responding with non-chaotic orbits of all kinds. For example, a damped harmonic regime is presented in Fig. 4 (for A = 0.47 and k = 4) as well as in Fig 5 (for k = 0 and k = 0.25). The periodic cycle is shown in Fig. 5 (k = 0.53). Finally, aperiodic divergent trajectories that indicate unbounded solutions of (6) are displayed in Fig. 5 (for k = 0.75 and bigger).

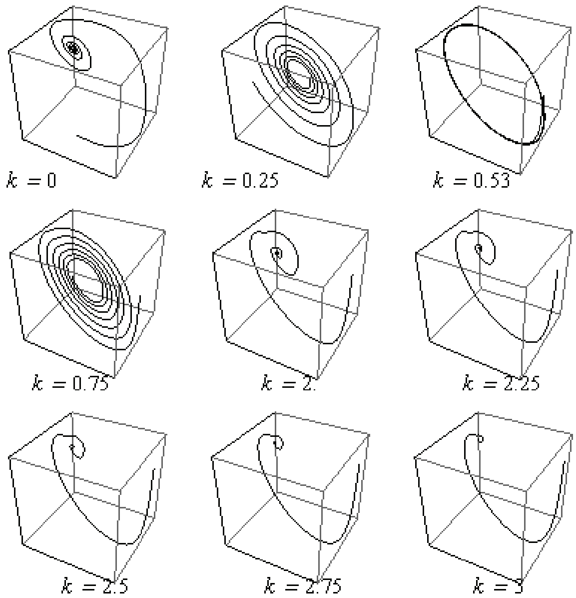

Fig. 5 A = 4.0, variable slope

4. Conclusion

One of the simplest dynamic flows that still exhibits chaotic behavior is Sprott's jerky system, given by equations (1) and (2) (see [8], [9]) i.e. $\dddot{x} = -A\ddot{x} + \varphi(\dot{x}) - x$, where $\varphi(\xi) = \xi^2$. This system "works" on the "edge" of chaos, which is evident from its small first Lyapunov exponent ($\lambda_1 = 0.0551$). Tracing for simpler function φ that yet supplies chaotic dynamics, Sprott and Linz [12] tried with $\varphi(\xi) = |\xi|$, the continuous function that represents a kind of piecewise linear approximation of quadratic function. In this case, no chaos has been detected.

The present note deals with the "hybrid" case embodied in two semi-quadratic functions, the left- and the right one, given by (7), see Fig. 1 for the graphs. Surprisingly, the left semi-quadratic function $g_k(\xi)$ produces chaos for some values of the larger part's variable slope k, and the corresponding value of A, while the symmetric, right semi-quadratic function $h_k(\xi)$ leads only to a non-chaotic dynamics.

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O JEDNOM "JERK" DINAMIČKOM SISTEMU

Ljubiša M. Kocić, Sonja Gegovska-Zajkova

Haotični sistemi koje je najpre u obliku elektronskih kola uveo J.C. Sprott [7-11] *pokazali su se kao zanimljivi primeri izvora slabog haosa – oblika haosa koji je jedino dozvoljen u osetljivim sistemima kao što su sistemi automatske kontrole i robotike. Ovde su razmatrane dve modifikacije 3D dinamičkog toka, koji je poznat kao jerk-dinamički sistem J.C. Sprott-a* [9].

Ključne reči: *"jerk"dinamika, haos, diferencijalne jednačine*