FACTA UNIVERSITATIS
Series: **Architecture and Civil Engineering** Vol. 2, N° 1, 1999, pp. 53 - 60

SECOND ORDER ANALYSIS OF SPACE BEAM STRUCTURES

UDC 624.072.2:519.6(045)

Marina Mijalković

Faculty of Civil Engineering and Architecture, Niš, Beogradska 14, Niš, YU

Abstract. *The subject of this paper is the analysis and design of space beam structures based on second-order theory using "exact" strain method. The advantage of this procedure over the "simplified" strain method is in including the influence of axial forces in the members on deformations of the structure, which is significant for some types of structures, such as: multistory frames, arches, suspension structures, etc.*
On the basis of theoretical consideration presented in this paper, the general program for PC has been developed. The program is intended for static analysis – the determination of nodal displacements and member forces of all types of space structures, as well as plane structures, which consist of members subjected to various external load. The first order analysis, as a spatial case of second order analysis is contained, too.
Using this program one suspension bridge was analyzed in the paper.

1. INTRODUCTION

The subject of this paper is the analysis and the calculation of effects in space linear structures based on the linearized Second Order Theory using the exact deformation method.

The Second Order Theory represents a special aspect of non-linear theory, where of the two basic assumptions of the generalized Non-linear Theory only one is adopted – the assumption that deflections and deflection deductions values are small so that their squares and higher grades can be neglected, on the basis of which linear connections between strains and deflections are obtained. However, the assumption that is not adopted is that deflections are so small that they can be neglected in equilibrium conditions, and because of that equations are non-linear. Thus, there are linear connections between strains and deflections, as well as between internal forces and strains, and non-linear connections between internal and external forces.

In this paper, the linearization of Second Order Theory equations is perfumed in this manner: values of axial forces are obtained in the first iteration on the basis of Second

Received August 10, 2000
Presented at 5th YUSNM Niš 2000, Nonlinear Sciences at the Threshold of the Third Millenium, October 2-5, 2000, Faculty of Mechanical Engineering University of Niš

Order Theory calculations, and not from the truss of the system as is done in approximate deformation method, while in the following iteration values of axial forces are calculated based on the Second Order Theory. Therefore, in respect to present-day theories, the additions made are that axial forces, introduced with the aim of linearization of the Second Order Theory equations, are calculated by the accurate deformation method, and that the influence of axial forces, which is significant in certain types of structures, such as shallow arches, suspension systems, multistory frames, upon the deformation of the system was taken into consideration.

The basic assumptions are that the material is ideally elastic, homogenous and isotropic, that the cross section is flat, non-deformable and normal onto the deformed beam axis after the bending. Beside the above Second Order Theory assumptions, the influence of transversal forces the strain is also neglected, while it is adopted the assumption that axial force is constant along the beam axis. Having in mind this assumption, in analyses of space linear structures subjected to several simultaneous types of stress – axial stress, multi-axial bending, torsion, it is possible to perform the superposition of these effects.

In the paper, the matrix analysis of structures is used, i.e. the Finite Element Method. The application of this method provides a simple presentation of mathematical model in a matrix form. Firstly, member stiffness matrix is formed, as a member is basic element in space structure; then, member stiffness matrices for all member boundary conditions were formed, based on the geometrically-static significance of its elements and the solutions of member differential Second Order Theory equations. Also, vectors for equivalent nodal forces were introduced due to member loading for all member boundary conditions.

Having in mind the basic relations for certain members, analogous reactions for the system consisting of two or more members were obtained. Already developed system stiffness matrices and equivalent loading vectors form a system of equations, whose solutions are deflections and node rotations, support reactions, and member internal forces.

This approach enabled the elaboration of a general computer program for the calculation of static and deformation effects using the Linearized Second Order Theory for all types of space beam structures, as well as plane structures. A short description of the programme is given and an example of calculation by applying this programme is enclosed.

2. DIFFERENTIAL EQUATIONS OF MEMBER BENDING IN SPACE PROBLEMS

The member element in space before and after the strain with internal forces and deflections in a local co-ordinate system is shown in Figure 1.

Considering separately the strains and deflections in xOy and xOz planes, pictures 2a and 2b, as the consequences of bending around the z-axis, i.e. the y-axis, the dependence between strains and deflections is obtained:

$$dx + du = (1+\varepsilon)dx\cos\varphi_z\,,\quad dv = (1+\varepsilon)dx\sin\varphi_z\,,\quad dw = (1+\varepsilon)dx\sin\varphi_y\,,$$

that is in agreement with the assumption on small scale strains, taking that $\cos\varphi_z \cong 1$, $\cos\varphi_y \cong 1$, $\sin\varphi_z \cong \varphi_z$, $\sin\varphi_y \cong \varphi_y$, $\varepsilon\varphi_z \cong 0$, $\varepsilon\varphi_y \cong 0$, the following is obtained:

$$du = \varepsilon dx\,,\quad dv = \varphi_z dx\,,\quad dw = \varphi_y dx\,.$$

Connections between internal and external forces on the basis of equilibrium conditions of a member in space after the strain, picture 1, with neglecting the squares and higher grades of deflections, are:

$$\frac{dH}{dx}=-p_x,\quad \frac{dV}{dx}=-p_y,\quad \frac{dW}{dx}=-p_z,\quad \frac{dM_x}{dx}-V\frac{dw}{dx}+W\frac{dv}{dx}=0,$$

$$\frac{dM_y}{dx}+H\frac{dw}{dx}-W\left(1+\frac{du}{dx}\right)=0,\quad \frac{dM_z}{dx}+H\frac{dv}{dx}-V\left(1+\frac{du}{dx}\right)=0.$$

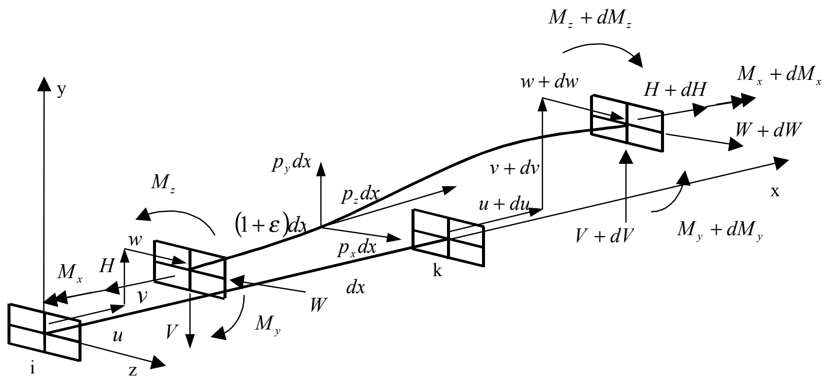

Fig. 1

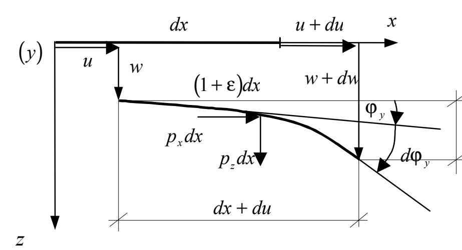

Fig. 2a

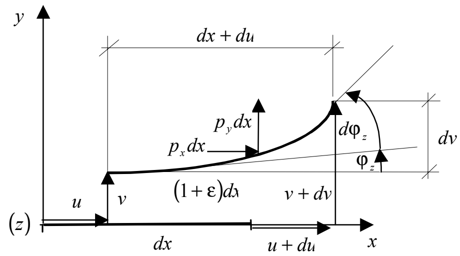

Fig. 2b

Connections between deflections and internal forces are:

$$\varepsilon=\frac{du}{dx}=\frac{N}{EA}+\alpha_t t,\quad \frac{d\varphi_z}{dx}=-\left(\frac{M_z}{EI_z}+\alpha_t\frac{\Delta t}{h}\right),\quad \frac{d\varphi_y}{dx}=-\left(\frac{M_y}{EI_y}+\alpha_t\frac{\Delta t}{b}\right).$$

A system of 12 equations with 12 unknowns can be simplified by neglecting the dilatation, and thus it boils down to:

$$\frac{du}{dx}=0,\ \varepsilon=\frac{N}{EA}+\alpha_T t=0,\quad \varphi_z=\frac{dv}{dx},\quad \varphi_y=\frac{dw}{dx},\quad \frac{dH}{dx}=-p_x,\quad \frac{dV}{dx}=-p_y,$$

$$\frac{dW}{dx}=-p_z,\quad \frac{dM_x}{dx}-V\varphi_y+W\varphi_z=0,\quad \frac{dM_y}{dx}-W+H\varphi_y=0,\quad \frac{dM_z}{dx}-V+H\varphi_z=0,$$

$$\frac{d\varphi_z}{dx} = -\left(\frac{M_z}{EI_z} + \alpha_t \frac{\Delta t}{h}\right), \quad \frac{d\varphi_y}{dx} = -\left(\frac{M_y}{EI_y} + \alpha_t \frac{\Delta t}{b}\right).$$

The linearization of the above system of non-linear equations is performed by determining the axial forces (H) in system members in the first iteration by the First Order Theory, while in the following iteration values for axial forces are calculated by the Second Order Theory.

Considering separately the bending in xOy and xOz planes, without regard to torsion, we obtain two different equations of a prismatic space member loaded with transverse load and axial force ($H = S = const$) at its ends in the form:

$$\frac{d^2}{dx^2}\left(EI_z \frac{d^2 v}{dx^2}\right) - \frac{d}{dx}\left(H \frac{dv}{dx}\right) = p_y - \frac{d^2}{dx^2}\left(EI_z \alpha_t \frac{\Delta t}{h}\right), \quad \frac{d^2}{dx^2}\left(EI_y \frac{d^2 w}{dx^2}\right) - \frac{d}{dx}\left(H \frac{dw}{dx}\right) = p_z - \frac{d^2}{dx^2}\left(EI_y \alpha_t \frac{\Delta t}{b}\right).$$

For the constant cross section member and constant temperature difference, this has the following form:

$$\frac{d^4 v}{dx^4} \pm k_z^2 \frac{d^2 v}{dx^2} = \frac{p_y}{EI_z}, \; k_z = \sqrt{\frac{|S|}{EI_z}}, \quad \frac{d^4 w}{dx^4} \pm k_y^2 \frac{d^2 w}{dx^2} = \frac{p_z}{EI_y}, \; k_y = \sqrt{\frac{|S|}{EI_y}},$$

where the plus symbol refers to the axial pressure force and minus to the axial tension force.

The solutions of differential equations 1,2 of space member in the case of the axial tension force, as well as rotations, bending moments and transversal forces expressions obtained by applying the beginning parameter method, are:

$$v(x) = v_0 + \varphi_{z0} \frac{\sin k_z x}{k_z} - M_{z0} \frac{1 - \cos k_z x}{S} - V_0 \frac{k_z x - \sin k_z x}{k_z S} + \int_0^x \frac{k_z(x-\xi) - \sin k_z(x-\xi)}{k_z S} p_y(\xi) d\xi,$$

$$\varphi_z(x) = v'(x) = \varphi_{z0} \cos k_z x - M_{z0} \frac{k_z \sin k_z x}{S} - V_0 \frac{1 - \cos k_z x}{S} + \int_0^x \frac{1 - \cos k_z(x-\xi)}{S} p_y(\xi) d\xi,$$

$$M_z(x) = -EI_z v''(x) = EI_z k_z \varphi_{z0} \sin k_z x + M_{z0} \cos k_z x + V_0 \frac{\sin k_z x}{k_z} - \int_0^x \frac{\sin k_z(x-\xi)}{k_z} p_y(\xi) d\xi,$$

$$V(x) = -EI_z v'''(x) - Sv'(x) = V_0 - \int_0^x p_y(\xi) d\xi$$

3. Matrix analysis of space linear structures

This paper analyzes the arbitrary space linear system formed of member elements, which is loaded with arbitrary load in system nodes and with concentrated, and distributed load in its members.

The dependence between the generalized member deflection vectors, as a basic finite element, and generalized member force vectors is given through a basic finite element equation:

$$\underline{R} = \underline{k}\underline{q} + \underline{Q}, \text{ where } \underline{R}^T = [N_i V_i W_i M_{xi} M_{yi} M_{zi} N_k V_k W_k M_{xk} M_{yk} M_{zk}],$$

$$\underline{q}^T = [u_i v_i w_i \varphi_{xi} \varphi_{yi} \varphi_{zi} u_k v_k w_k \varphi_{xk} \varphi_{yk} \varphi_{zk}] \text{ and } \underline{Q}^T = [Q_1 Q_2 Q_3 Q_4 Q_5 Q_6 Q_7 Q_8 Q_9 Q_{10} Q_{11} Q_{12}]$$

is the equivalent member loading vector, which represent concentrated loading at member ends which substitutes external effect acting along the member axis.

Member stiffness matrix $\underline{k}$ is a matrix by means of which a direct connection between the basic static and strain member values is set up, and for the member in space it is of order 12 and it is symmetrical with the main diagonal line, which is the consequence of Maxwell's statement on the mutuality of deflections. Beside member type **k**, members with different boundary conditions were also processed, and corresponding stiffness matrices were worked out.

Equivalent loading vector components of external loading in *y* and *z* directions in the form of a concentrated force, of partially uniform distributed loading and partially linear distributed loading, were determined for various types of pressed and tensed members by means of applying the Linearized Second Order Theory.

Starting from basic relations for a single member, analogous relations for the system consisting of two or more members are obtained. The system stiffness matrix is formed from a single member stiffness matrices, and the system equivalent loading vector is formed from single member equivalent loading vectors and from loading directly acting in nodes. In this way a system of equations is obtained, on the basis of which node deflections and rotations, support reactions, and finally internal member forces are determined. It is necessary to define the position of each member in the global coordinate system, the transformation of generalized force vectors, generalized deflection vectors, as well as the transformation of member stiffness matrix from the local into the global coordinate system should also be performed. This connection is expressed by a matrix equation:

$$\underline{q} = \underline{\lambda}\underline{q}^{*} ,$$

where $\underline{\lambda}$ is a transformation matrix obtained by translating the global coordinate system deflection vectors to local coordinate system axes directions, whose coefficients represent angle cosinuses between the global and local coordinate system axes, $\underline{q}$ is deflection vector in the local co-ordinate system and $\underline{q}^{*}$ is deflection vector in the global coordinate system.

Generalized deflections, generalized forces and equivalent loading vectors are transformed from the local into the global one by inverse transformation:

$$\underline{q}^{*} = \underline{\lambda}^{T}\underline{q} ,$$

where

$$\underline{\lambda}^{T} = \underline{\lambda}^{-1} ,$$

which means that the transformation matrix is orthogonal. Generalized forces and equivalent loading vectors are transformed in the same manner.

When these expressions are substituted with a basic finite element equation, the expression for member stiffness matrix in the global co-ordinate system is obtained.

$$\underline{k}^{*} = \underline{\lambda}^{T}\underline{k}\underline{\lambda} .$$

For every element of **j** of the structure, which is influenced by external loading connections between generalized forces and generalized deflections in the global coordinate system is:

$$\underline{R}_j^* = \underline{k}_j^* \underline{q}_j^* + \underline{Q}_j^* , \quad j = 1,2,...,M$$

where M is the total number of system members.

When the previous equation is combined for all system members, matrix equation of the whole structure is obtained in this form:

$$\underline{\overline{R}}^* = \underline{\overline{K}}^* \underline{\overline{q}}^* + \underline{\overline{Q}}^* ,$$

where $\underline{\overline{R}}^*$, $\underline{\overline{q}}^*$, $\underline{\overline{Q}}^*$ and $\underline{\overline{K}}^*$ are generalized forces, generalized deflections, equivalent deflections and unrelated members stiffness matrices vectors.

Structure members are mutually connected and they must satisfy the compatibility conditions in system nodes, so that the member connection matrix or the system compatibility matrix is introduced, named $\underline{A}$.

Beside conditions for compatibility in system nodes, equilibrium conditions must be also satisfied. Nodes are also affected by external loading so that equilibrium conditions for all system nodes can be represented by a matrix equation:

$$\underline{P}^* - \underline{R}^* = 0 ,$$

where $\underline{P}^*$ is vector of given external loading and $\underline{R}^*$ vector of connection forces in system nodes.

Using the principle of virtual deflections, a matrix equation is obtained:

$$\underline{R}^* = \underline{A}^T \underline{\overline{R}}^* ,$$

and by further changings, we obtain:
$\underline{K}^* \underline{q}^* = \underline{S}^*$, (1), where $\underline{K}^* = \underline{A^T \overline{K}^* A},$ is system stiffness matrix, and $\underline{S}^* = \underline{P}^* - \underline{Q}^*$, free members vector.

Member deflection and rotations are determined from equation (1), but it is first necessary to define structure support conditions.

If free deflections and rotations are grouped and shown as vector components $\underline{q}_s^*$, and known deflections and rotations of support members as components $\underline{q}_o^*$, equation system (1), can be represented in this form:

$$\begin{bmatrix} \underline{K}_{ss}^* & \underline{K}_{so}^* \\ \underline{K}_{os}^* & \underline{K}_{oo}^* \end{bmatrix} \begin{bmatrix} \underline{q}_s^* \\ \underline{q}_o^* \end{bmatrix} = \begin{bmatrix} \underline{S}_s^* \\ \underline{S}_o^* \end{bmatrix} ,$$

i.e. it can be separated into two equation systems. The first equation system provides structure node deflections and the second provides support reactions.

4. COMPUTER PROGRAM FOR STATIC ANALYSIS OF SPACE LINAR SYSTEMS APPLYING THE SECOND ORDER THEORY

Theoretical assumption given in previous chapters enabled the elaboration of STABIL-M computer program which is made to look like the well-known STRESS program. It enables the analysis of space and plane linear systems using the linearized Second Order Theory. It is written in FORTRAN computer language and adapted for PC use.

Incoming information and testing of their logic comprises is the first part of the program.

The second part of the program solves the problem using the First Order Theory by the accurate deformation method until static and deformation values are obtained. With member axial forces obtained in this way, stiffness matrix for each member using the Second Order Theory is formed, i.e. modified stiffness matrix depending on whether the releases of structure members are performed, equivalent loading vector members are calculated using the Second Order Theory and condition equations of accurate deformation methods are solved.

Outcoming results are: static and strain values using the First Order Theory and static and strain values using the Second Order Theory.

5. CALCULATION EXAMPLE – SUSPENSION BRIDGE

This example represents the calculation of effects in a suspension bridge with a span of 132 m (Figure 3), loaded with a non-symmetrical continual loading. As in these structures the horizontal tension is very big, the influence of these forces upon bending moments values is very significant. The application of STABIL-M program gave results which showed that deformation values (deflection v) are smaller using the Second Order Theory for even 238%, for φ_z rotation it is 207%, and M_z bending moments 87%. This raises the doubt of whether linear theory is rightfully applied because much lesser M_z values obtained using the Second Order Theory indicate that it represents a more economical procedure.

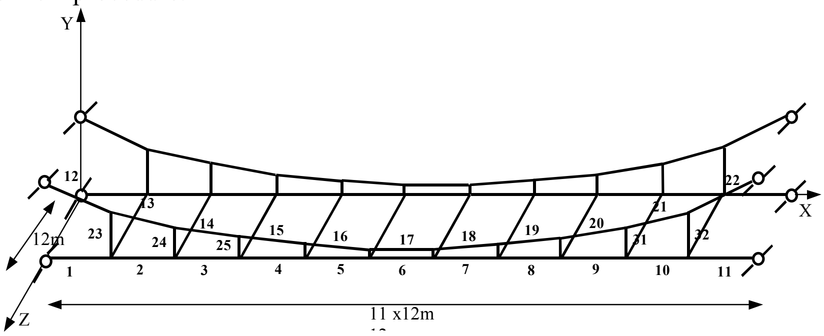

Fig. 3.

6. CONCLUSION

This paper deals with the application of Finite elements method upon the calculation of space linear structures using the linearized Second Order Theory. Axial forces effect upon the deformations, which is in some structure types very significant, is encompassed, so that it can be said that the calculation is worked out by applying the accurate deformation method using the linearized Second Order Theory. STABIL-M program made on the basis of theoretical assumptions that are stated in the paper was tested on a number of models and calculation results were compared with programs known and adopted worldwide.

The application of STABIL-M program, i.e. the linearized Second Order Theory, in some structure types, such as shallow arches, suspension structures and very tall, thin constructions, gives considerable differences in results using the First and the Second Order Theories respectively, which points out the justifiableness of the Second Order Theory application in design of some structure types. A more accurate calculation of shallow arches gives much greater moment values which means that the results obtained using the Second Order Theory are more applicable for the structure safety, while a more accurate calculation of a suspension bridge (Figure 3) gives much smaller moment results which means that the application of Second Order Theory gives more economical solutions, a lot of savings is made in the material, and thus the construction becomes thinner and more beautiful.

The advantage of STABIL-M program is that it is universal, applicable in all sorts of plane and space linear structures, and adopted for easy usage. Unlike this program, in most other programs structure modeling, mathematical model adoption, structure data input and outcoming data interpreting represents a big problem, so that STABIL-M program is recommended for wide use.

REFERENCES

1. M.Sekulović – *Geomerijski nelinearna analiza linijskih nosača*, Teorija konstrukcija, Savremeni problemi nelinearne analize, pp43-89, Građevinska knjiga, Beograd, 1992.
2. M.Stanković – *Proračun okvira po Teoriji drugog reda tačnom metodom deformacije*, Savez društva građevinskih konstruktera Jugoslavije, VII kongres, Cavtat, 1983.
3. B.Stanković – *Sračunavanje uticaja po Teoriji drugog reda primenom tačne metode deformacije*, Naše građevinarstvo 44, pp158-162, 1990.

PRILOG PRORAČUNU KONSTRUKCIJA PRIMENOM TEORIJE DRUGOG REDA

Marina Mijalković

Predmet ovog rada je analiza i proračun uticaja kod prostornih linijskih nosača po linearizovanoj Teoriji drugog reda po tačnoj metodi deformacije. Prednost ovog postupka u odnosu na približnu metodu deformacije je što obuhvata i uticaj normalnih sila na deformaciju sistema, koji je značajan kod nekih konstrukcija kao što su višespratni ramovi, lukovi, lančani sistemi i dr.

Na bazi teorijskih razmatranja prikazanih u radu razvijen je jedan generalni program za proračun statičkih i deformacijskih veličina po linearizovanoj teoriji drugog reda svih tipova linijskih nosača u prostoru, kao i u ravni. Uključena je i teorija prvog reda, kao specijalni slučaj teorije drugog reda.

U radu je analiziran primer lančanog mosta koji je primenom ovog programa obrađen.