FACTA UNIVERSITATIS
Series: **Architecture and Civil Engineering** Vol. 2, N° 1, 1999, pp. 31 - 40

GENERALISATION OF SPHERE POLARITY TO CONTOUR LINE DETERMINATION AND SHADING OF SURFACES OF REVOLUTION

UDC 514.144:514.181.2:519.688(045)

Radovan Štulić, Zoran Milojević, Jugoslav Bajkin

Faculty of Engineering, Novi Sad

Abstract. *The paper deals with the problem of realistic representation of surfaces of revolution. Boundary representation of surfaces (the most common way used in descriptive geometry) comprises that the surface is represented by its visible edges including contour lines when curved surfaces are taken into account. The image of a surface becomes more realistic if it is presumed that the surface is illuminated by certain source of light (either lamp or sun). This is particularly important for curved surfaces such as a surface of revolution since the illumination of its points varies all over the surface. In this paper we have shown that the same method can be applied for determination of contour line, illumination dividing line (further IDL) and iso-illuminated lines (further isophots) when both parallel projecting rays and parallel light rays are taken into account. The method is based upon the generalisation of the polarity of spheres to the polarity of surfaces of revolution. An algorithm for computer aided computation and drawing of such surfaces/solids has been created as well.*

Key words: *surface of revolution, polarity, representation, oblique projection, illumination*

1. INTRODUCTION

Among the problems of realistic surface representation, the contour line determination takes an important place. Once the contour line determined the visible part of the surface from its invisible part is directly separated. No more visibility determination of points of the surface is required so that the use of standard methods of computer graphics can be avoided. Knowing the contour line, which is a view dependent property of the surface, the data storage of the surface is made easier and less demanding. Further more the boundary representation which is normally used in descriptive geometry, which is in computer graphics regarded as inconvenient (because of its complexity) for curved surface, can now been used for surface of revolution representation.

Received September 25, 2000

The problem of contour line determination is already known and developed in numerous investigations. It is usually based upon the tangent planes determination of the particular position using methods of differential geometry. On one hand it can be regarded as a general method but on the other it might sometimes appear difficult to apply.

Whenever a descriptive geometric approach can be applied to certain problem of surface representation, the solution of the problem turns out to be easier and more convenient for computer aided use [2] and [7].

In this paper, based upon the results given in [2], [7] and [8], a pure analytical procedure has been developed for determination of contour line, illumination dividing line and iso-illuminated lines when surface of revolution is represented in oblique projection and at the same time illuminated by parallel light rays. The method used in the procedure is based upon the generalisation of the polarity of sphere to the polarity of surface of revolution and is unique for all three topics. Some illustrative examples are presented as well.

2. Geometric structure of surfaces of revolution

The representation of an object may appear as a need for representation of either already existing or "new" one. In case of the latter, one may almost always use mathematical description of these objects. When the matter is on the surfaces of revolution, usual way of its modelling is by revolving two-dimensional entity about an axis in space [5].

In this paper we used the following notation: vectors or vector functions are represented by bold letters f. e. **i**; **f(t)** and first derivatives with respect to certain parameter **t** by $\dot{(\sim)} = d(\sim)/dt$.

Therefore, surface of revolution **R** can be generated by rotating a plane curve **f**, called meridian, about its coplanar axis o. Let the meridian be given

$$\mathbf{f}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k};\quad t \in [t_0, t_1]\,, \tag{1}$$

where the condition of coplanarity is valid:

$$A\,x(t) + B\,y(t) + C\,z(t) + D = 0\,, \tag{2}$$

and the axis of revolution by its vector $\mathbf{o} = (o_x, o_y, o_z)$, and one point on it $O = (O_x, O_y, O_z)$.

Since the contour line of a surface is only a view dependent line, in this paper, for the simplicity of the obtained formulae, we shall presume that the axis of revolution is coincident with one coordinate axis (in this paper z axis) and its coplanar meridian lies in one coordinate plane (Oyz plane). Therefore we have the meridian

$$\begin{aligned} \mathbf{f}(t) &= y(t)\,\mathbf{j} + z(t)\,\mathbf{k}; \\ x(t) &\equiv 0 \\ t &\in [t_0, t_1] \end{aligned} \tag{3}$$

and the axis o of its vector $\mathbf{o} = (0, 0, 1)$, through the coordinate origin.

The surface of revolution generated by revolving the meridian Eq. 3. about the axis has its vector equation:

$$\mathbf{R}(t,\theta) = y(t)\cos\theta\,\mathbf{i} + y(t)\sin\theta\,\mathbf{j} + z(t)\,\mathbf{k};\quad t \in [t_0, t_1];\quad \theta \in [0, 2\pi]. \tag{4}$$

This means that certain point $F(0, y(t_F), z(t_F))$ lying on the meridian Eq. 3. rotates for the given angle θ about the axis *o* into the point $F_R\,(y(t_F)\cos\theta_{F_R}, y(t_F)\sin\theta_{F_R}, z(t_F))$.

The equations of meridians and parallels of the surface Eq. 4. are obtained when taking $t = const.$ and $\theta = const.$, respectively.

Meridians and parallels form an orthogonal net of curves on the surface of revolution. The tangent plane to the surface is plane determined by tangents to the meridian and parallel.

3. DETERMINATION OF CONTOUR LINE AND ILLUMINATION DIVIDING LINE

It is known that the contour line of any curved surface is a space curve lying on the surface, which is in fact an envelope of points of tangency of tangent planes to the surface which pass through the centre of projection (both in perspective and parallel projection). In the same way, illumination-dividing line of any curved surface is the envelope of points tangency of tangent planes to the surface, which pass through the light source being either at finiteness or infinity.

3.1. Family of Auxiliary Touching Spheres

Let us introduce a family of spheres given by the equation

$$f(X,Y,Z,t) = X^2 + Y^2 + (Z - \zeta(t))^2 - R(t)^2 \tag{5}$$

where $t \in [t_0, t_1]$ is a parameter and terms ζ(t) and *R*(t) are given by the following expressions:

$$\zeta(t) = z(t) + \frac{\dot{y}(t)}{\dot{z}(t)};\quad R(t) = y(t)\sqrt{1 + \left(\frac{\dot{y}(t)}{\dot{z}(t)}\right)^2} \tag{6}$$

expressing the condition that each sphere of the family Eq. 5. has one parallel of tangency with the surface Eq. 4. in common, that is, each sphere is a touching sphere of the surface of revolution.

3.2. Polar plane of sphere

It is known [6] that, for certain ideal point, the polar plane of a sphere intersects the sphere at its great circle (through the centre of the sphere). It means that in the case parallel projecting rays whose unit vector is $\mathbf{p} = (p_x, p_y, p_z)$ and light source lying at infinity given by the unit vector of light $\mathbf{s} = (s_x, s_y, s_z)$ polar planes of each sphere have the following equations

$$p_x\tilde{x} + p_y\tilde{y} + p_z(\tilde{z} - \zeta(t)) = 0 \tag{7}$$

$$s_x\tilde{x} + s_y\tilde{y} + s_z(\tilde{z} - \zeta(t)) = 0 \tag{8}$$

In the case of light, it can be noted that the angle between tangent planes to each point of the circle of the sphere lying in the polar plane Eq. 8. and the light rays is equal to zero. This means that each point has a zero illumination. Points that lie on the illuminated part of the sphere are illuminated according to the angle between tangent planes to the sphere at that point and the light rays (incident angle). If the following presumptions are taken into account:

1. light source is isotropic emitting the light to all directions,
2. the medium of the light transmitting is homogeneous and isotropic,
3. surfaces are non-transparent and ideal mirrors
4. influence of reflected light on the illumination is neglected and
5. magnitude of the light does not diminish by the distance,

it is not difficult to prove that points of the sphere that have the same illumination lie on a circle of the sphere which is parallel to the polar plane for infinitely distant light source (Fig. 1). The equation of this plane is as follows:

$$s_x \overset{*}{x} + s_y \overset{*}{y} + s_z\left(\overset{*}{z} - \zeta(t) - \frac{R(t)\sin\alpha}{s_z}\right) = 0\,. \tag{9}$$

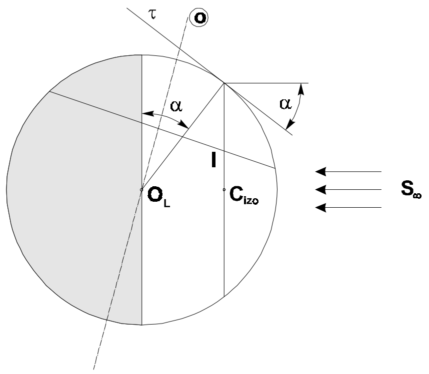

Fig. 1. Sphere with polar planes and planes of iso- illuminated

3.3. Polar cylinders

Polarity of the family of auxiliary touching spheres can be generalised onto the polarity of its surface of revolution. The equivalent to the polar plane of sphere is a polar cylinder of surface of revolution since each sphere from the family touches the surface at one parallel. Namely if polar planes Eq. 7 and Eq. 8 are intersected by the plane of their common parallel equations of two lines are obtained, first as a contour line and second as an IDL:

$$p_x\tilde{X} + p_y\tilde{Y} + p_z y(t)\frac{\dot{y}(t)}{\dot{z}(t)} = 0;\ \tilde{Z} = z(t)\,, \tag{10}$$

$$s_x\hat{X} + s_y\hat{Y} + s_z y(t)\frac{\dot{y}(t)}{\dot{z}(t)} = 0;\ \hat{Z} = z(t)\,. \tag{11}$$

Each equation can be written in the canonical form

$$\frac{\tilde{X}}{p_y} = \frac{\tilde{Y} - \frac{p_z}{p_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)}}{-p_x} = \frac{\tilde{Z} - z(t)}{0} = \nu_p , \tag{10a}$$

$$\frac{\hat{X}}{s_y} = \frac{\hat{Y} - \frac{s_z}{s_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)}}{-s_x} = \frac{\hat{Z} - z(t)}{0} = \nu_p , \tag{11a}$$

or in the parametric form:

$$\begin{aligned} \tilde{X}(t,\nu_p) &= p_y \nu_p \\ \tilde{Y}(t,\nu_p) &= \frac{p_z}{p_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} - p_x \nu_p , \\ \tilde{Z}(t,\nu_p) &= z(t) \end{aligned} \tag{10b}$$

$$\begin{aligned} \hat{X}(t,\nu_s) &= s_y \nu_s \\ \hat{Y}(t,\nu_s) &= \frac{s_z}{s_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} - s_x \nu_s , \\ \hat{Z}(t,\nu_s) &= z(t) \end{aligned} \tag{11b}$$

which, more obviously, represents a straight-line generated surface.

3.4. Contour line and illumination dividing line

The intersecting curve between the surface of revolution Eq. 4. and its polar cylinders Eq. 10. and Eq. 11. is, first a contour line and second ILD, whose 3D coordinates are given as follows:

$$\begin{aligned} \tilde{X}_{C_1/C_2}(t) &= p_y \nu_{p\text{-}C_1/C_2} \\ \tilde{Y}_{C_1/C_2}(t) &= \frac{p_z}{p_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} \text{-} p_x \nu_{p\text{-}C_1/C_2} , \\ \tilde{Z}_{C_1/C_2}(t) &= z(t) \end{aligned} \tag{12}$$

$$\begin{aligned} \hat{X}_{ILD_1/ILD_2}(t) &= s_y \nu_{s\text{-}ILD_1/ILD_2} \\ \hat{Y}_{ILD_1/ILD_2}(t) &= \frac{s_z}{s_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} - s_x \nu_{s\text{-}ILD_1/ILD_2} , \\ \hat{Z}_{ILD_1/ILD_2}(t) &= z(t) \end{aligned} \tag{13}$$

where

$$\nu_{p-C_1/C_2} = \frac{p_x p_z}{p_y (p_x^2 + p_y^2)} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} \pm \frac{y(t)}{p_x^2 + p_y^2} \sqrt{p_x^2 + p_y^2 - \left(p_z \frac{\dot{y}(t)}{\dot{z}(t)} \right)^2} , \tag{14}$$

$$\nu_{s-ILD_1/ILD_2} = \frac{s_x s_z}{s_y (s_x^2 + s_y^2)} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} \pm \frac{y(t)}{s_x^2 + s_y^2} \sqrt{s_x^2 + s_y^2 - \left(s_z \frac{\dot{y}(t)}{\dot{z}(t)} \right)^2} \,, \tag{15}$$

which being projected by parallel projecting rays give generally an oblique projection of the surface of revolution with ILD This is, in fact, the transformation of 3D into 2D coordinates onto the frontal projection plane i.e. Oyz plane. The transformation formulae are taken from [7]:

$$\begin{aligned} y_{2D} &= y_{3D} - \lambda\, x_{3D} \cos \varphi \\ z_{2D} &= z_{3D} - \lambda\, x_{3D} \sin \varphi \end{aligned} \,, \tag{16}$$

where λ and φ are "shortening" coefficient and angle $<(-xOy)$, respectively, usual parameters of an oblique projection in descriptive geometry. The relation between the vector of projection rays **p** and λ and φ are:

$$\begin{aligned} p_x &= 1 \\ p_y &= \lambda \cos \varphi \\ p_z &= \lambda \sin \varphi \end{aligned} \,. \tag{17}$$

3.5. Visibility test

The contour line divides a surface of revolution on its visible and invisible part. Thus, points on each parallel of the surface are either on the visible part or invisible, where the boundary points are contour points that lie on the contour line – space curve of the surface. In order to determine which points of the parallel are on the visible side of the surface we have performed simple visibility test according to which an arbitrary point N of the parallel together with each of boundary – contour points C_1 and C_2 forms two vectors $\mathbf{NC_1}$ and $\mathbf{NC_2}$, whose sum is a vector directed out of the surface. Thus a scalar product between this sum and rays of projection gives:

$$\begin{aligned} (\mathbf{NC_1} + \mathbf{NC_2}) \cdot \mathbf{p} > 0 \;&\Rightarrow\; \textit{point N lies on invisible side of the surface} \\ (\mathbf{NC_1} + \mathbf{NC_2}) \cdot \mathbf{p} < 0 \;&\Rightarrow\; \textit{point N lies on visible side of the surface} \end{aligned}$$

4. SHADING BASED UPON ON ISO-ILLUMINATED LINES

Iso-illuminated lines, being loci of points of the same illumination gives the base for exact either shading or colouring of the visible and illuminated parts of surfaces. These lines, not belonging to the net of meridians and parallels of a surface of revolution, can be determined in the same way as contour line and ILD, when equivalents of polar cylinders are introduced.

4.1. Equivalents of polar cylinders of surfaces of revolution

If, instead of polar plane for the infinitely distant light source, circles of iso-illuminated lines are treated for each sphere, equivalent cylinders to polar cylinder are obtained Fig. 2. Let us analyse such cylinder that will give points on the surface whose illumination is proportional to the incident angle 0<α<90. In this case each parallel of the surface intersected by the plane of the iso-illuminated points of the sphere Eq. 9. produces again a cylinder of generatrices which are parallel to generatrices of polar cylinder but at certain distance. The equation of this cylinder (here called equivalent to the polar cylinder) is as follows:

$$s_x \overset{*}{X} + s_y \overset{*}{Y} + s_z y(t)\frac{\dot{y}(t)}{\dot{z}(t)} + R(t)\sin\alpha = 0; \quad , \qquad \overset{*}{Z} = z(t) \tag{18}$$

canonical

$$\frac{\overset{*}{X}}{s_y} = \frac{\overset{*}{Y} - \dfrac{s_z}{s_y} y(t)\dfrac{\dot{y}(t)}{\dot{z}(t)} + \dfrac{R(t)\sin\alpha}{s_y}}{-\mathrm{s_x}} = \frac{\overset{*}{\mathrm{Z}} - \mathrm{z(t)}}{0} = \nu_{s\text{-}iso}\,, \tag{18a}$$

parametric

$$\begin{aligned} &\overset{*}{X}(t,\nu_{s\text{-}iso}) = s_y \nu_{s\text{-}iso} \\ &\overset{*}{Y}(t,\nu_{s\text{-}iso}) = \frac{s_z}{s_y} y(t)\frac{\dot{y}(t)}{\dot{z}(t)} + \frac{R(t)\sin\alpha}{s_y} - s_x \nu_{s\text{-}iso} \quad . \\ &\overset{*}{Z}(t,\nu_{s\text{-}iso}) = z(t) \end{aligned} \tag{18b}$$

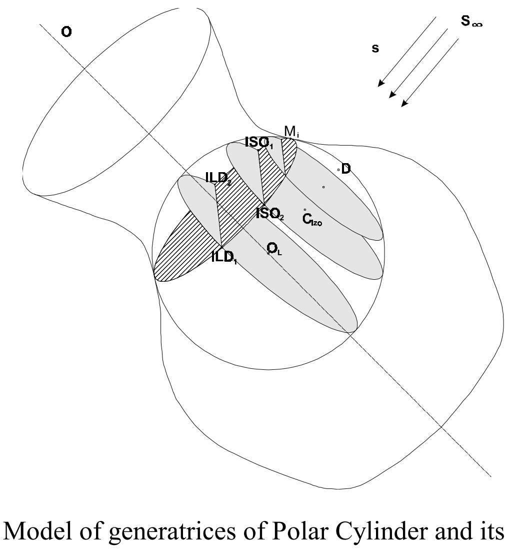

Fig. 2. Model of generatrices of Polar Cylinder and its Equivalents

4.2. Iso-illuminated lines – isophots

Family of equations of isophots on the surface of revolution are obtained as an intersection of the surface Eq. 4. and its induced cylinder Eq. 18 by the light source and incident angle α:

$$\begin{aligned} &\overset{*}{X}_{s-iso_1/iso_2}(t) = s_y \nu_{s-iso_1/iso_2} \\ &\overset{*}{Y}_{s-iso_1/iso_2}(t) = \frac{s_z}{s_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} + \frac{R(t)\sin\alpha}{s_y} - s_x \nu_{s-iso_1/iso_2} , \\ &\overset{*}{Z}_{s-iso_1/iso_2}(t) = z(t) \end{aligned} \tag{19}$$

where

$$\nu_{s-iso_1/iso_2} = \frac{\rho(t,\alpha)\, s_x}{s_x^2 + s_y^2} \pm \frac{\sqrt{y^2(t)\,(s_x^2 + s_y^2) - \rho^2(t,\alpha)\, s_y^2}}{s_x^2 + s_y^2} , \tag{20}$$

and

$$\rho(t,\alpha) = \frac{s_z}{s_y} y(t) \frac{\dot{y}(t)}{\dot{z}(t)} + \frac{R(t)\sin\alpha}{s_y} , \ \alpha \in [0, \alpha_{\max} \le 90^0] . \tag{21}$$

Since not every point of each iso-illuminated line lies on the visible part of the surface the visibility test has to be applied.

5. Examples

According to the previous we have created the following algorithm:

```
Program SURFACE OF REVOLUTION;
begin
   INPUT:     meridian, axis, unit vector of
              projecting rays and light;
   CALCULATE:3D of contour line, ILD, isophots
   TRANSFORM:3D to 2D coordinates
   DRAW:      contour line,
              ILD,
              colouring between isophots;
              edges of lower and upper basis
              after having performed visibility test
end.
```

Fig. 3. and Fig. 4. illustrate a surface of revolution in oblique projection first when unit vectors **p** and **s** are different and second when they coincide.

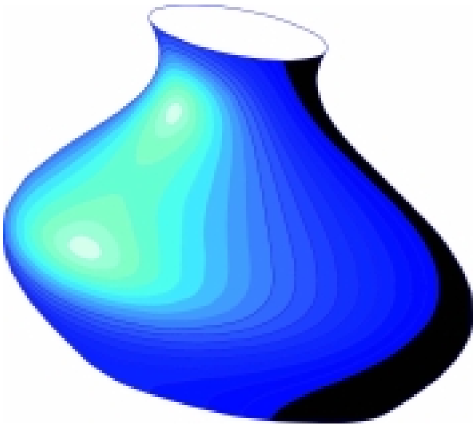

Fig. 3.

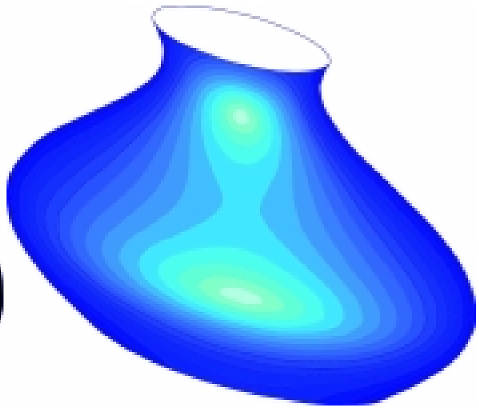

Fig. 4.

6. CONCLUSION

In the paper we have developed same analytical procedure for direct determination of contour line, illumination dividing line (ILD) and iso-illuminated lines of surfaces of revolution. The procedure is based upon descriptive geometric method of auxiliary touching spheres by generalising the polarity of spheres onto the polarity of surfaces of revolution, that is, each surface of revolution generates polar cylinder with respect to the given centre of projection or light source whose intersection is a space curve, lying on the surface, i.e. contour line. In the same way iso-illuminated lines have been treated. The restriction we have taken into account is that the surface does not hide itself, which means that each parallel the surface of revolution has real contour points. More general case of self-hiding surfaces is to be analysed in further investigations.

REFERENCES

1. Foley, J., van Dam, A., Feiner, S. and Hughes, J.: *Computer Graphics-Principles and Practise*, Addison Wesley Publishing company Inc., 1996.
2. Mihajlovic Levi, V. and Stulic, R.: *Surfaces of Revolution in CAD: Contour Line Determination*, Proceedings of the 5th ASEE International Conference on Engineering Computer Graphics and Geometry, Melbourne, Australia, 1992. Vol. 1, pp. 358-362.
3. Pejovic, T.: *Mathematical Analysis V* (in Serbian language), Naucna knjiga Beograd, 1978.
4. Prvanovic, M.: *Projective Geometry* (in Serbian language), Naucna knjiga Beograd, 1986.
5. Rogers, F. D. and Adams, A. J.: *Mathematical Elements for Computer Graphics*, McGraw Hill International Editions, 1990.
6. Sbutega, V.: *Synthetic Geometry III* (in Serbian language), Postgraduate Course, University of Beograd, 1986.
7. Stulic, R.: *Descriptive Geometric Methods in Computer Graphics: Contour Lines and Isophots of Surfaces of Revolution* (in Serbian language), Master Thesis, Beograd, 1994.
8. R., Jandric, Z. and Milojevic Z.: *Polar Cylinders of Surfaces of Revolution: Contour Line Determination*, XIII Yugoslav seminar on Geometry, Novi Sad, 1998.
9. Wheatherburn, C. E.: *Differential Geometry of Three Dimensions*, Cambridge University Press, 1955.

GENERALIZACIJA SFERNOG POLARITETA NA DETERMINACIJI KONTURNE LINIJE I SENČENJU OBRTNIH POVRŠI

Radovan Štulić, Zoran Milojević, Jugoslav Bajkin

Rad se bavi problemom realistične reprezentacije obrtne površi.Granična reprezentacija površina (najuobičajeniji način koji se koristi u deskriptivnoj geometriji) podrazumeva da je površina predstavljena svojim vidljivim ivicama, uključujući i konturne linije kada se u obzir uzmu i zakrivljene površi. Slika površine postaje realističnija ako se pretpostavi da je površina obasjana nekim izvorom svetlosti (lampom ili suncem). Ovo je pogotovo važno za zakrivljene površi kao što su obrtna površ budući da osvetljenost njenih tačaka varira po celoj površini. U ovom radu pokazali smo da isti metod može biti primenjen za determinaciju konturne linije, razdelne linije osvetljenosti (u daljem tekstu RLO) i izo-osvetljenih linija (u daljem tekstu izofoti), kada se u obzir uzmu i paralelni projektujući zraci i paralelni svetlosni zraci. Ovaj metod je baziran na generalizaciji polariteta sfera na polaritet obrtnih površi. Urađen je i algoritam za kompjutersko izračunavanje i crtanje ovakvih površina.

Ključni pojmovi: *obrtna površ, polaritet, predstavljanje, kosa projekcija, osvetljenost.*