FACTA UNIVERSITATIS
Series: **Architecture and Civil Engineering** Vol. 2, N° 1, 1999, pp. 15 - 21

ANALYSIS OF BENDING OF SURFACES USING PROGRAM PACKAGE *MATHEMATICA*

UDC 514.772.35(045)

Ljubica S. Velimirović

Faculty of Science, Department of Mathematics, University of Niš
Ćirila i Metodija 2, 18000 Niš, Yugoslavia
E-mail: vljubica@archimed.filfak.ni.ac.yu

Abstract. *In this paper we consider surface bending using program package Mathematica. Bending and infinitesimal bending of surfaces are considered. Infinitesimal bending of rotational surfaces is considered using Cohn-Vossen's method. The results obtained by solving differential equations are presented graphically.*
As an example of non-rigid surfaces Belov's surface is pointed out. Belov's non-rigid toroid is drawn at Mathematica. The surface that is infinitesimal bending of the Belov's toroid is also given. Field of infinitesimal bending for this surface is also considered. The joint circles generated by the apices of Belov's quadrangle, deformed to the curves that are not circles are also given.

Key words: *bending, infinitesimal bending, infinitesimal bending field, toroid*

INTRODUCTION

Bending of surfaces is a kind of deformations of surfaces under which a surface is included in a family of isometric surfaces. Characteristic well-known example is bending from helicoid to catenoid. In fact, these two surfaces are the beginning and the end of a deformation through isometric minimal surfaces [3]. This deformation is given using symbolic manipulation program *Mathematica* (Fig. 1).

Infinitesimal bending of surfaces is a kind of deformations of surfaces at R^3 under which arc length of curve on the surface is stationary. The surface S is included in a family of surfaces S_ε, $\varepsilon \in [0,1)$ so that $ds_\varepsilon - ds = o(\varepsilon)$, or $\delta s = 0$, i.e. arc length has zero variation.

Infinitesimal bending of surfaces is the subject of the investigation both at the field of building mechanics and also in the theory of deformation of surfaces, a part of differential geometry. Infinitesimal bending of surfaces is relatively rarely considered using computer graphics. It is certainly the case that computers can be effective supplement to pure

Received February 6, 2001

thought, because the graphs they can make provide insights into complex relationships.

We will here use *Mathematica* to draw accurate pictures of infinitesimal bending of surfaces. The results obtained at [1], [4] and [5], made "by hand" will be used here. We will enlarge them here with some more calculations.

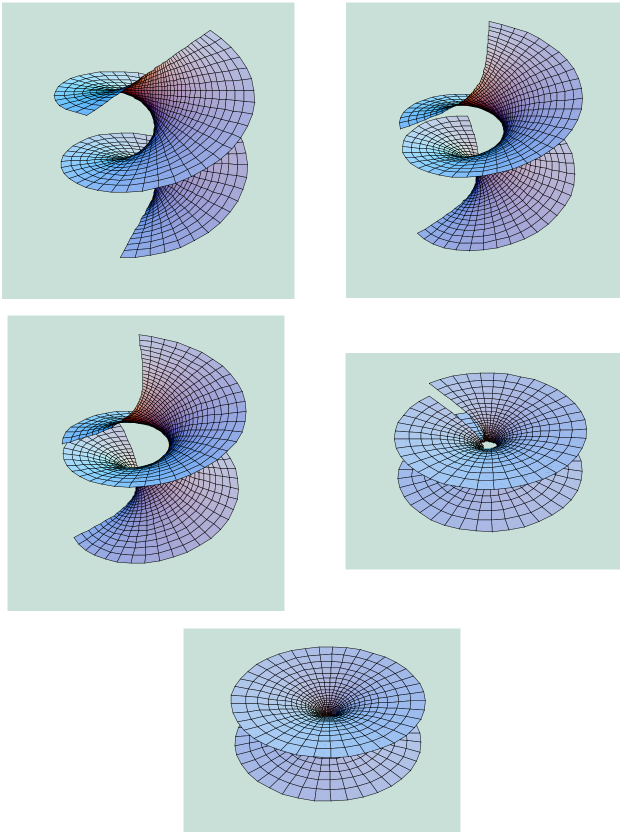


Fig. 1.

1. Infinitesimal Bending of Rotational Surfaces

Many works at the theory of deformations of surfaces are dedicated to the problem of infinitesimal bending of rotational surfaces. The problem of generating the field of infinitesimal bending is for this class of surfaces reduced to the problem of solving the system of ordinary differential equations. We will consider infinitesimal bending of surfaces using Cohn-Vossen’ s method, which is the basic for rotational surfaces.

The special coordinate system with the orthonormal base $\bar{e},\bar{a}(v),\bar{a}'(v)$ is introduced. The axe of rotation has the ort $\bar{e}$, and ort $\bar{a}(v) \perp \bar{e}$ at O. The relation between initial base and the new one is

$$\bar{e}=\bar{k},\bar{a}(v)=\cos v\bar{i}+\sin v\bar{j},\bar{a}'(v)=-\sin v\bar{i}+\cos v\bar{j} .$$

At $(\overline{e}, \overline{a})$ plane we consider a curve C which is meridian of rotational surface S, C: $\rho = \rho(u)$. The equation of the surface generated by revolving of the curve C is:

$$S : \overline{r}(u,v) = u\ \overline{e}\ + \rho(u)\ \overline{a}(v). \tag{1.1}$$

Infinitesimal bending field of the surface S, $\overline{z}(u,v)$ is

$$\overline{z}(u,v) = \alpha(u,v)\ \overline{e}\ + \beta(u,v)\ \overline{a}(v) + \gamma(u,v)\ \overline{a}\ '(v). \tag{1.2}$$

The next theorem is in force:

Theorem 1.1. *Necessary and sufficient condition for a field* $\overline{z}(u,v)$ *to be a field of infinitesimal bending of rotational surface S is that the functions* $\alpha(u,v)$, $\beta(u,v)$, $\gamma(u,v)$ *satisfy the system of partial differential equations*

$$\begin{gathered}\alpha_u(u,v) + \rho'(u)\ \beta_u(u,v) = 0,\\ \beta(u,v) + \gamma_v(u,v) = 0,\\ \alpha_v(u,v) + \rho'(u)\,(\beta_v(u,v) - \gamma(u,v)) + \rho(u)\ \gamma_v(u,v) = 0.\end{gathered} \tag{1.3}$$

The functions $\alpha(u,v)$, $\beta(u,v)$, $\gamma(u,v)$, are periodical functions with respect to v with period 2π. They can be developed in Fourier series:

$$\begin{aligned}\alpha(u,v) &= \sum_{k=-\infty}^{+\infty} e^{ikv}\ \varphi_k(u),\\ \beta(u,v) &= \sum_{k=-\infty}^{+\infty} e^{ikv}\ \psi_k(u),\\ \gamma(u,v) &= \sum_{k=-\infty}^{+\infty} e^{ikv}\ \chi_k(u),\end{aligned} \tag{1.4}$$

where $\varphi_k(u)$, $\psi_k(u)$, $\chi_k(u)$, $k \in Z$, are complex functions. The field $\overline{z}(u,v)$ can be written as

$$\overline{z}(u,v) = \sum_{k=-\infty}^{+\infty} e^{ikv}[\varphi_k(u)\overline{e} + \psi_k(u)\overline{a}(v) + \chi_k(u)\overline{a}'(u)] = \sum_{k=-\infty}^{+\infty} e^{ikv}\overline{z}_k(u,v) \tag{1.5}$$

where

$$\overline{z}_k(u,v) = [\varphi_k(u)\overline{e} + \psi_k(u)\overline{a}(v) + \chi_k(u)\overline{a}'(u)]e^{ikv}, \quad k \in Z \tag{1.6}$$

The next theorem is in force [2]:

Theorem 1.2. *The functions* $\varphi_k(u)$, $\psi_k(u)$, $\chi_k(u)$, $k \in Z$, *satisfy the next system of differential equations*

$$\begin{gathered}\varphi'_k(u) + \rho'(u)\psi_k(u) = 0, \psi_k(u) + ik\chi_k(u) = 0,\\ ik\varphi_k(u) + \rho'(u)[ik\psi_k(u) - \chi_k(u)] + \rho(u)\chi_k{}'(u) = 0,\end{gathered} \tag{1.7}$$

and the functions $\psi_k(u)$, $\chi_k(u)$ *satisfy the same differential equation of the second order*

$$\rho(u)\psi_k{}''(u) + (k^2 - 1)\rho''(u)\psi_k(u) = 0\ , \tag{1.8}$$

or

$$\rho(u)\chi_k{}''(u) + (k^2 - 1)\rho''(u)\chi_k(u) = 0. \tag{1.8'}$$

2. INFINITESIMAL BENDING OF BELOV'S TOROID

It is known that circular torus is rigid. Among the surfaces that are topologically equivalent to torus, K. M. Belov pointed to the class of non-rigid surfaces. At [1] K. M. Belov considered infinitesimal bending of a toroidal surface of revolution, generated by revolving of the meridian in the shape of a special quadrangle, and among them pointed to the class of non-rigid surfaces with convex meridian. The class of non-rigid surfaces topologically equivalent to the torus was enlarged at [6] with toroidal surfaces with polygonal meridian. We will here consider non-rigid toroids according to [4].

Theorem (K. M. Belov [1]) *The quadrangle* **B***, with apexes* $A(-1,b)$, $B(0,b+c_1)$, $C(1,b)$, $D(0,b-c_2)$, *rotates around u-axis of the coordinate system* $u\,O\,\rho$. *Necessary and sufficient condition for the non-rigidity of the toroid rotational surface generated by the meridian* **B** *is*

$$\frac{1}{c_2}-\frac{1}{c_1}=\frac{k^2}{b}, \quad (k\in N, k\geq 2) \tag{2.1}$$

b, c_1, c_2 *are positive parameters.*

Belov's surface consists of cones generated by the sides of Belov's quadrangle. Revolving AB generates $S_{(1)}$, and by revolving BC, CD, DA , the parts $S_{(2)}$, $S_{(3)}$, $S_{(4)}$, are generated respectively. The equations of the Belov's surface are

$$S_{(1)}:\bar{r}_{(1)}(u,v)=\rho_{(1)}(u)\cos v\bar{i}+\rho_{(1)}(u)\sin v\bar{j}+u\bar{k}=$$
$$[b+c_1(u+1)]\cos v\bar{i}+[b+c_1(u+1)]\sin v\bar{j}+u\bar{k}, \quad u\in[-1,0], \quad v\in[0,2\pi],$$

$$S_{(2)}:\bar{r}_{(2)}(u,v)=[b+c_1(1-u)]\cos v\bar{i}+[b+c_1(1-u)]\sin v\bar{j}+u\bar{k}, \quad u\in[0,1], v\in[0,2\pi]$$

$$S_{(3)}:\bar{r}_{(3)}(u,v)=[b+c_2(u-1)]\cos v\bar{i}+[b+c_2(u-1)]\sin v\bar{j}+u\bar{k}, \quad u\in[0,1], v\in[0,2\pi]$$

$$S_{(4)}:\bar{r}_{(4)}(u,v)=[b-c_2(u+1)]\cos v\bar{i}+[b-c_2(u+1)]\sin v\bar{j}+u\bar{k}, \quad u\in[-1,0], v\in[0,2\pi]$$

At the figure 2. Belov's toroid is given. In this case $k = 2$, $b = 1$, $c_1 = 1/2$, $c_2 = 1/6$.

Deformed surface S, S_ε, for the field $\bar{z}(u,v)$, has the equation

$$S_\varepsilon:\bar{r}(u,v,\varepsilon)=\bar{r}(u,v)+\varepsilon\bar{z}(u,v). \tag{2.2}$$

For the field $\bar{z}(u,v)$, according to [4], we have at the part of Belov's surface generated by AB:

$$Z_{(1)}:\bar{z}_{(1)}(u,v)=[2(u-P)M_1\cos kv\cos v-2(P-u)\frac{M_1}{k}\sin kv\sin v]\bar{i}$$
$$+[2(u-P)M_1\cos kv\sin v+2(P-u)\frac{M_1}{k}\sin kv\cos v]\bar{j}$$
$$-2c_1M_1u\cos kv\bar{k}, u\in[-1,0], v\in[0,2\pi],$$

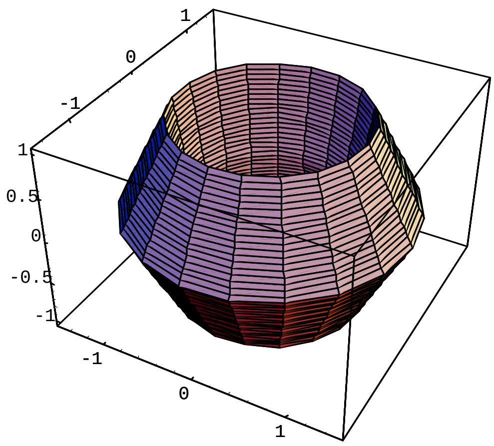
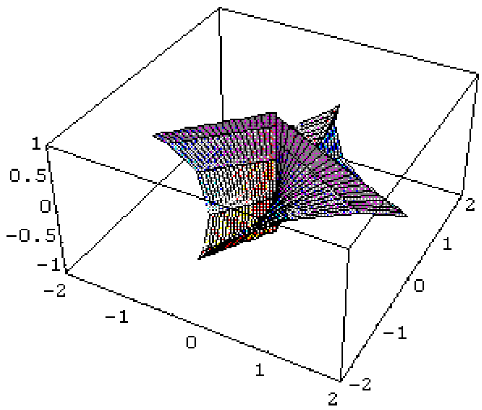


Fig. 2.

Fig. 3. shows this infinitesimal bending field.



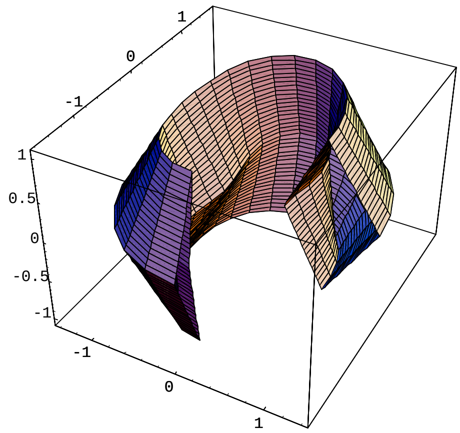


Fig. 3. Fig. 4.

Deformed Belov's toroid is S_ε (Fig. 4.), and the parts of it are $S_{(i)\varepsilon}$, $i = 1,2,3,4$. We can finally determine the toroid that is infinitesimal bending of the Belov's toroid

$$S_\varepsilon = \bigcup_{i=1}^{4} S_{(i)\varepsilon}\,, \quad S_{(i)\varepsilon} : \bar{r}_{(i)\varepsilon} = \bar{r}_{(i)}(u,v) + \varepsilon \bar{z}_{(i)}(u,v)$$

$$\begin{aligned} S_{(1)\varepsilon} = \{[b + c_1(u+1)]\cos v + 2\varepsilon(u-P)M_1(\cos kv \cos v + \frac{1}{k}\sin kv \sin v)\}\bar{i} \\ + \{[b + c_1(u+1)]\sin v + \varepsilon 2(u+P)M_1(-\cos kv \sin v + \frac{1}{k}\sin kv \cos v)\}\bar{j} \\ + [u + \varepsilon(-2c_1 M_1 u \cos kv)]\bar{k}, u \in [-1,0], v \in [0,2\pi] \end{aligned}$$

$$S_{(2)\varepsilon} = \{[b + c_1(1-u)]\cos v - 2\varepsilon(u+P)M_1(\cos kv \cos v + \frac{1}{k}\sin kv \sin v)\}\bar{i}$$
$$+\{[b + c_1(1-u)]\sin v + \varepsilon 2(u+P)M_1(-\cos kv \sin v + \frac{1}{k}\sin kv \cos v)\}\bar{j}$$
$$+[u + \varepsilon(-2c_1 M_1 u \cos kv)]\bar{k}, u \in [0,1], v \in [0,2\pi],$$

Figure 5. shows the parts $S_{(3)\varepsilon}, S_{(4)\varepsilon}$ of the bent toroid for $\varepsilon = 0.1$ and Figure 6. the parts $S_{(4)}, S_{(4)\varepsilon}$

$$S_{(3)\varepsilon} = \{[b + c_2(u-1)]\cos v + 2\varepsilon(u-Q)M_1(\cos kv \cos v + \frac{1}{k}\sin kv \sin v)\}\bar{i}$$
$$+\{[b + c_2(u-1)]\sin v + \varepsilon 2(u-Q)M_1\frac{c_1}{c_2}(\cos kv \sin v - \frac{1}{k}\sin kv \cos v)\}\bar{j}$$
$$+[u + \varepsilon(-2c_1 M_1 u \cos kv)]\bar{k}, u \in [0,1], v \in [0,2\pi]$$

$$S_{(4)\varepsilon} = \{[b - c_2(u+1)]\cos v - 2\varepsilon(u+Q)M_1\frac{c_1}{c_2}(\cos kv \cos v + \frac{1}{k}\sin kv \sin v)\}\bar{i}$$
$$+\{[b - c_2(u+1)]\sin v + \varepsilon 2(u+Q)M_1\frac{c_1}{c_2}(-\cos kv \sin v + \frac{1}{k}\sin kv \cos v)\}\bar{j}$$
$$+[u + \varepsilon(-2c_1 M_1 u \cos kv)]\bar{k}, u \in [-1,0], v \in [0,2\pi]$$

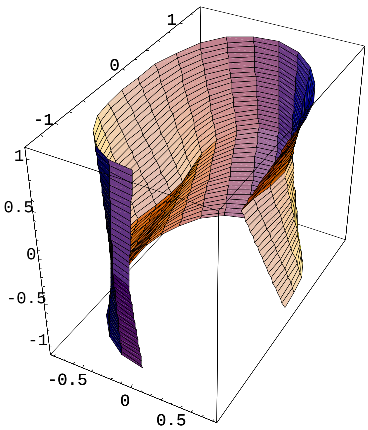


Fig. 5.

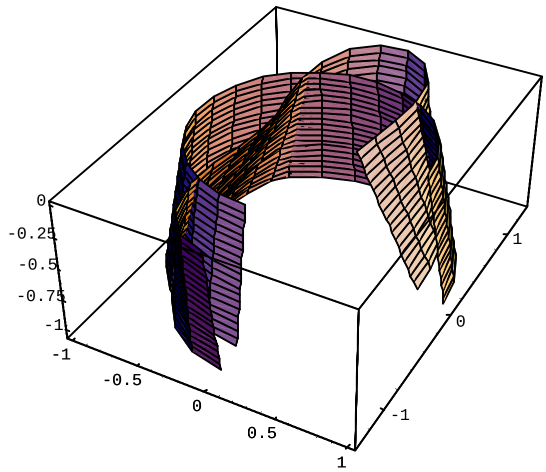


Fig. 6.

The apeces of the Belov's quadrangle circumscribe the circles revolving around the axes. Both this circles and the curves that are infinitesimal bending of them are given on the Figure 7.

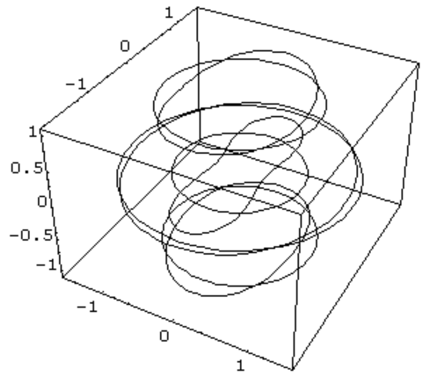


Fig.7.

REFERENCES

1. Belov K. M., *O beskonechno malyh izgib. torobraznoi pov. vrashceniya*, Sib. Mat. Zh.t. IX, No3 (1968), 490-494.
2. Cohn-Vossen, S. E., *Nekotorye voprosy differ. geometrii v celom*, Fizmatgiz, Moskva, 1959.
3. Gray A., *Modern differential geometry of curves and surfaces*, CRC Press,1993.
4. Velimirović Lj. S., *A new proof of a theorem of Belov, Publications of the Inst. Math*. (Beograd) (NS) 63 (77) (1998), 102-115.
5. Velimirović Lj. S., *On variation of the volume under infinitesimal bending of a closed rotational surface*, Novi Sad J. Math., vol. 29, No.3, 1999, 377-386.
6. Velimirović Lj. S., *Beskonechno malye izgibaniya toroobraznoi poverhnosti vrascheniya s mnogougolnym meridianom*, Izv. VUZ Mat., 1997. No. 9, 3-7.

ANALIZA SAVIJANJA POVRŠI UZ KORIŠĆENJE PROGRAMSKOG PAKETA *MATHEMATICA*

Ljubica S. Velimirović

U radu se razmatra savijanje površi uz korišćenje programskog paketa Mathematica. Predstavljen je poznati primer savijanja katenoida do helikoida . U radu se razmatra beskonačno malo savijanje rotacionih površi uz korišćenje metoda Kon-Fosena. Rezultat dobijen rešavanjem diferencijalnih jednačina prikazan je grafički.

Specijalno, nacrtan je primer nekrutog toroida Belova . Polje savijanja dato je grafički, a takodje i deformisani toroid,kako po delovima, tako i u celini. Krugovi spajanja koji su generisani temenima Belovljevog četvorougla deformisani u krive koje nisu krugovi su takodje dati.

Ključne reči: *savijanje, beskonačno malo savijanje, toroid, polje beskonačno malog savijanja*