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CRITERIA FOR CONSTRUCTIVE DERIVATION OF THE REAL DOUBLE POINTS OF THE BICIRCULAR 4TH DEGREE CURVES

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Abstract. *This paper analyses the criteria which can be used to determine the bicircular curves of the* **4th** *degree. Constructive derivation of the selected curves is accomplished by the quadratic transformation in the corresponding transformation model. By adjusting the performances of the model, it is achieved that the presented bicircular curves do not have real infinitely distant points. The analysed conditions under which, by the further specialisation of the accepted transformation model, the real double points of those curves are analysed, their number and characteristics determined: the shape of a loop or a pointed top.*

Key words: *bicircular curves, double points, polarity, pencil of circumferences, involuntary series.*

INTORODUCTION

Two dominant characteristics determine the study and classification of the **4th** degree curves: behaviour of curves in the infinity, or the reality and multiplicity of the infinitely distant points and the nature and the number of the double points.

Bicircular **4th** degree curves make a characteristic group of curves whose double, imaginary conjugated, points are coinciding with the absolute points of the plane.

There are several ways for the constructive derivation of these curves, depending on their genesis:

- stereographic projection of the tracing curved sphere and the **2nd** degree plane, which does not pass through the projection centre, inversion of the circular **3rd** degree curve, where the focuses of the curve correspond to the focuses of the resulting **4th** degree curve,
- quadratic transformation of the **2nd** degree curves in the co-ordinate triangle – which yields rational curves.
- other characteristic constructions of the specific families of the bicircular **4th** degree

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curves, Descartes oval, Cassinian oval, Persey's curves, which carry the names of the mathematicians who invented, designed and studied them.

1. CONSTRUCTIVE DERIVATION OF THE BICIRCULAR 4TH DEGREE CURVES

The method for the derivation of the bicircular **4th** degree curves, used in this paper, is based on the previous surveys published in the papers /5/, /6/, /7/ and comprises the quadratic transformation of the **2nd** degree series which is realised in the pencil of the polar fields, which result is the general **4th** degree curve. The general model of transformation is given by the series of **2nd** degree poles on the conic **k** and two conics, $\mathbf{k_1}$ and $\mathbf{k_2}$, which, as the basic conics of the pencil, determine the pencil of the polar fields. The choice of the conic **k**, $\mathbf{k_1}$ and $\mathbf{k_2}$ characteristics and their inter-relations represents the specialisation of the transformation model on which the conditions and the result of the transformation (the characteristics of the resulting **4th** degree curve) are dependent.

The choice that the result of transformation should be the bicircular **4th** degree curve, calls for the primary specialisation of the model: giving **k**, $\mathbf{k_1}$ and $\mathbf{k_2}$ conics, as well as **K**, $\mathbf{K_1}$ and $\mathbf{K_2}$ circumferences, **K**, $\mathbf{K_1}$ and $\mathbf{K_2}$. A transformation model adopted in this way makes possible the constructive derivation of the bicircular **4th** degree curves, because the circumferences **k**, $\mathbf{k_1}$ and $\mathbf{k_2}$ pass through the absolute points of the plane.

The selected group of the bicircular curve family, whose constructive derivation is presented in the examples **I**, **II**, **III**, **IV**, has no real infinitely distant points, which requires additional adjustments in the adopted model, which should be in accordance to the criteria for determination of the infinitely distant points (dealt with in the papers /5/, /6/, /7/): circumference **k** has no common points with the straight $\mathbf{K_1K_2}$, which is the centre of the pencil. (fig. 1., 2., 3., 4. and 5.).

The basic procedures of the constructive derivation of the bicircular curve are presented by the selected examples and they consist of: polar transformation, joining and intersecting of the appended elements.

By the polar transformation (copying) of the series of **2nd** points (poles) on the **k, 1**,**2**,**3**… circumference, in relation to the basic circumference $\mathbf{k_1}$, on the basis of polarity characteristics, as the projective correlation, we get a projectively appended pencil of the **2nd** degree Σ_1 **1**,**2**,**3**… The series of poles **k**, **1**,**2**,**3**…, in respect to the basic circumference $\mathbf{k_2}$, are projectively appended by the Σ_2 pencil of the $\overline{\mathbf{1}},\overline{\mathbf{2}},\overline{\mathbf{3}}$, … polars in the same way.

The Σ_1 and Σ_2 pencils are projectively appended to the same series of poles **k**, and so are mutually. That is the first step of the quadratic transformation. In the second step, by intersecting the couples of appended polars, $\mathbf{1x\overline{1}}$, $\mathbf{2x\overline{2}}$, $\mathbf{3x\overline{3}}$,… of the pencils Σ_1 and Σ_2, we determine the points of the **4th** degree curves (fig.1.–5.).

As the circumferences, **k**, $\mathbf{k_1}$, $\mathbf{k_2}$, which pass through the absolute points of the plane, hence the absolute points are a pair of poles of **k** series, which transforms, by means of the polarity, into the pairs of isotropic tangents i.e. the polars of $\mathbf{k_1}$ and $\mathbf{k_2}$ circumferences. The absolute points and their isotropic tangents are mutually dual and incident elements in respect to $\mathbf{k_1}$ and $\mathbf{k_2}$ circumferences. By intersecting of the appended pairs of isotropic tangents, two pairs of auto-intersecting, imaginary conjugated points of the derived curve, coinciding with the absolute point, on which basis is proven that the resulting **4th** degree curve passes through the absolute points of the plane two times, which makes it bicircular.

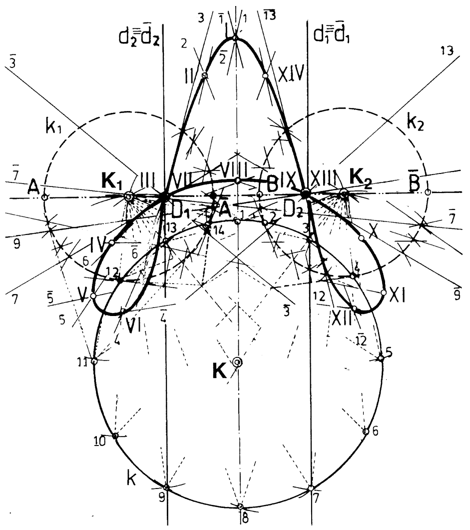

Fig. 1.

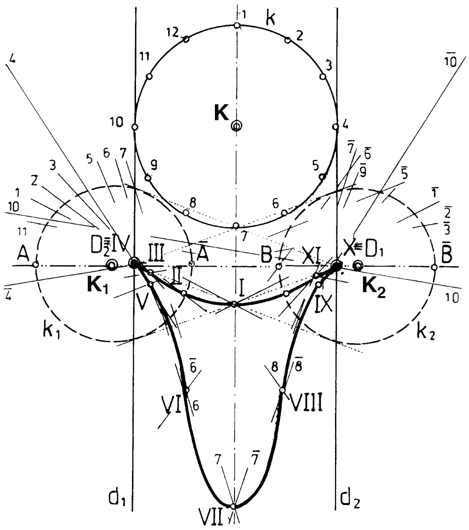

Fig. 2.

2. DOUBLE ELEMENTS OF THE CIRCUMFERENCE PENCIL POLAR FIELDS

Every conic induces the involution of points on the straight line. The pencil of a conic is, by any straight line in the plane, intersected across colocal involuntary series, whose pairs of appended points, on the selected straight line, are determined by the intersection of every pencil circumference with that straight line.

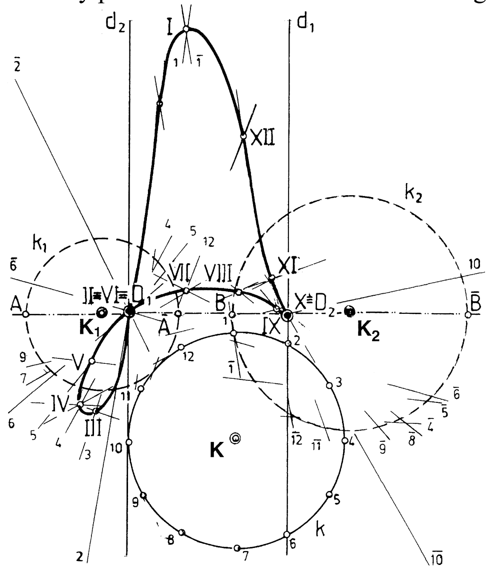

Fig. 3.

The involuntary series on the central $\mathbf{K_1}$ and $\mathbf{K_2}$ are observed, of the circumference pencils determined by the basic circumferences $\mathbf{k_1}$ and $\mathbf{k_2}$ form the adopted transformation model. The pairs of appended points $\mathbf{A\overline{A}}$ and $\mathbf{B\overline{B}}$, determine the colocal involution on the central, induced by its pencil of circumferences. At the appropriate circumference pencils, this involution can by hyperbolic, parabolic and elliptical.

We find the double points of the involuntary series on the central in the hyperbolic and parabolic pencils of circumferences.

In the adopted model, on the selected examples **I-V** (fig. 1.–5.) we analyse the hyperbolic pencils of circumferences, determined by the basic circumferences $\mathbf{k_1}$

and $\mathbf{k_2}$, of the $\mathbf{K_1}$ and $\mathbf{K_2}$ centres. $\mathbf{D_1}$ and $\mathbf{D_2}$ double points, of the involuntary series on the centrals of the pencils, are determined by the pairs of appended points $\mathbf{A}\overline{\mathbf{A}}$ and $\mathbf{B}\overline{\mathbf{B}}$, by using the Steiner's circumference.

In the pencil of the polar fields, determined by the circumference pencils, the $\mathbf{D_1}$ point which is a pole, is appended polar $\mathbf{d_1}$ through the point $\mathbf{D_2}$ in respect to the circumference $\mathbf{k_1}$, while $\mathbf{D_1} \equiv \overline{\mathbf{D}_1}$ pole is appended polar $\mathbf{d_2} \equiv \overline{\mathbf{d}_2}$ in respect to the circumference $\mathbf{k_2}$. The mutually dual correspondence $\mathbf{D_2} \equiv \overline{\mathbf{D}_2}$, is determined in the same way, with $\mathbf{d_1} \equiv \mathbf{d}_1$, in respect to the basic circumferences of the pencil, $\mathbf{k_1}$ and $\mathbf{k_2}$.

The $\mathbf{D_1} \equiv \overline{\mathbf{D}_1}$ and $\mathbf{D_2} \equiv \overline{\mathbf{D}_2}$, points, and $\mathbf{d_1} \equiv \overline{\mathbf{d}_1}$ and $\mathbf{d_2} \equiv \overline{\mathbf{d}_2}$ straight lines represent the double elements of the polar fields of the circumference pencils which dualise the incidence: $\mathbf{D_1} \equiv \overline{\mathbf{D}_1}$ is incident with $\mathbf{d_2} \equiv \overline{\mathbf{d}_2}$, as well as $\mathbf{D_2} \equiv \overline{\mathbf{D}_2}$ with $\mathbf{d_1} \equiv \overline{\mathbf{d}_1}$.

3. Real double points of the bicircular 4^th^ degree curves

The involuntary correlation in the adopted model executes the transformation of the polars of all the points of the $\mathbf{d_1}$ straight line in projectively appended polar pencil through $\mathbf{D_1}$ point in respect to the basic circumference $\mathbf{k_1}$. Analogous to this, the polars of all the points of the $\mathbf{d_1} \equiv \overline{\mathbf{d}_1}$ straight form the projectively appended polar pencil through $\mathbf{D_1} \equiv \overline{\mathbf{D}_1}$ point, in respect to the basic circumference $\mathbf{k_2}$.

On the basis of the previous interpretation and the characteristics of the double elements of the hyperbolic pencils of the polar fields of the circumferences, determined in the previous chapter, we can define the terms for obtaining the double points of the bicircular $\mathbf{4^{th}}$ degree curve as well as the interpretation of its characteristics. These terms are determined by the relation of the double straight lines of the polar field pencil to the **k** circumference. There are the following possibilities:

A) the double straight has two real, separated common points with the **k** circumference ($\mathbf{d_1} \equiv \overline{\mathbf{d}_1}$, or $\mathbf{d_2} \equiv \overline{\mathbf{d}_2}$ intersects **k**, i.e. k circumference approaches the double straight from two sides),

B) the double straight line and **k** circumference have two real coinciding common points ($\mathbf{d_1} \equiv \overline{\mathbf{d_1}}$ is tangent to **k** – **k** circumference approaches the double straight from the same side),

C) the double straight line has no common points with the **k** circumference ($\mathbf{d_1} \equiv \overline{\mathbf{d}_1}$ is a passant of the **k** circumference).

Two common points of the double straight line and the **k** circumference are two poles whose pair of appended polars, in respect to both of the basic circumferences, $\mathbf{k_1}$ and $\mathbf{k_2}$, passes (intersects) the corresponding double point.

The result is that one double point of the resulting bicircular curve coincides with the double point of the hyperbolic pencil of the polar fields of the circumferences, on condition that the **k** circumference has two common points with any other double straight line of this pencil of the polar fields. The double point has the characteristics of a node (loop), if the A) condition is satisfied, or of a pointed top, if the B) condition is satisfied.

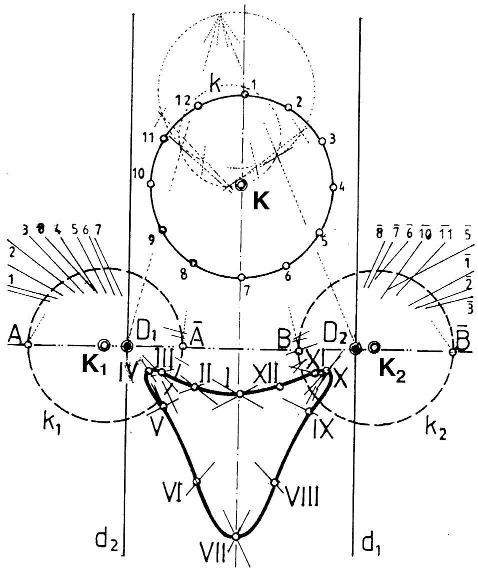

Fig. 4.

3.1. Two real separeted double points of the bicirkcular 4th degree curve

The hyperbolic pencils of the polar fields of the circumferences have two double straight lines, $\mathbf{d_1}$ and $\mathbf{d_2}$, which can intersect **k** circumference in two pairs of correspondent poles. We conclude that, in this adopted transformation model, one can get two real, separated double points of the bicircular **4th** degree curve.

The example I (fig.1.) displays that both of the straight lines, of the determined hyperbolic pencil of the polar fields intersect the k circumference, one across the **3** and **7** pair of poles, the other across the **9** and **13** pair of poles. That is how two real, separated, double points, $\mathbf{III} \equiv \mathbf{VII} \equiv \mathbf{D_1} \equiv \overline{\mathbf{D}_1}$ and $\mathbf{IX} \equiv \mathbf{XIII} \equiv \mathbf{D_2} \equiv \overline{\mathbf{D}_2}$ are obtained.

The **k** circumference approaches the intersection points **3**, **7** and **9**, **13** from the different sides of the double straight lines, so the double points of the bicircular **4th** degree curve in the form o a loop (node).

The example II (fig.2.) analyses the different relation of the double straight lines of the hyperbolic pencil of the polar fields to the series of poles on the **k** circumference. The double straight lines are tangent to the **k** circumference. In the points of tangency **4** and **10**, the **k** circumference and the corresponding double straight line have a pair of real, coinciding poles. The **k** circumference approaches the double points of tangency from the same sides of the double straight lines. That satisfies the conditions for determination of two real, separated, double points, $\mathbf{IV} \equiv \mathbf{D_2} \equiv \overline{\mathbf{D}_2}$ and $\mathbf{X} \equiv \mathbf{D_1} \equiv \overline{\mathbf{D}_1}$, of the bicircular **4th** degree curve, both in the shape of pointed top.

The example III (fig.3.) displays the further possibilities for the specialisation within the transformation model, where one of the double lines has two real separated points – poles **2** and **6** with the **k** circumference, while the other double straight line has two real common points – poles, coinciding in the point **10**. The bicircular curve resulting under these conditions has two real, separate, double points. As the **k** circumference approaches the **2** and **6** points from the different sides, one of the double points of the curve $\mathbf{II} \equiv \mathbf{VI} \equiv \mathbf{D_1} \equiv \overline{\mathbf{D}_1}\ \overline{\mathbf{D}_1}$ is in the form of a loop, and the other, $\mathbf{X} \equiv \mathbf{D_2} \equiv \overline{\mathbf{D}_2}$, of a pointed top.

The example IV (fig.4.) illustrates the case when the **k** circumference and the double straight lines of the hyperbolic pencil of the polar fields have no common points. In addition, the necessary conditions A) or B) for the specialisation of the transformation model, which realise the possibility for obtaining the double points, are not satisfied. Constructive derivation of bicircular **4th** degree curve has no two real double points.

3.2. Two real coinciding double points of the bicircular 4th degree curves

The parabolic pencil of the polar fields of the circumferences, in the adopted transformation model, is determined with two basic circumferences $\mathbf{k_1}$ and $\mathbf{k_2}$, which are mutually tangent in two doubly coinciding points, $\overline{\mathbf{A}} \equiv \mathbf{B} \equiv \mathbf{D_1} \equiv \mathbf{D_2}$, of the parabolic involuntary series on the central, whose pairs of appended points are $\mathbf{A}\overline{\mathbf{A}}$ and $\mathbf{B}\overline{\mathbf{B}}$. This point as a pole, in respect to the both basic circumferences has a correspondent polar line $\mathbf{d_1} \equiv \mathbf{d_2}$, in which two double straight lines of the pencil coincide.

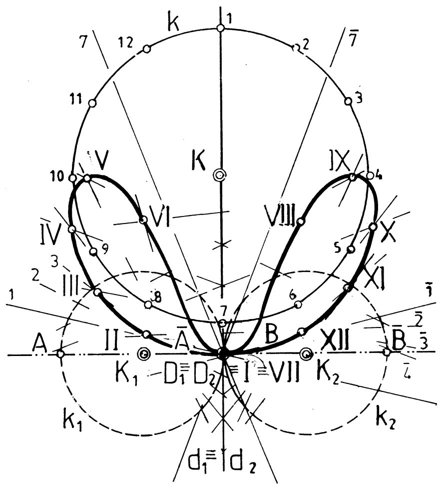

Fig. 5.

The example V on the fig.5. determines the specific conditions for the constructive derivation of the bicircular **4th** degree curve with two real, coinciding double points. The relation of the double straight line of the parabolic straight line of the polar fields to the **k** circumference, within the adopted model, defines the position of the poles **1** and **7** on the k circumference, which, by the transformation, give double points $\mathbf{II} \equiv \mathbf{VII} \equiv \mathbf{D_1} \equiv \mathbf{D_2}$.

4. CONCLUSION

Bicircular **4th** degree curves, by the definition, have two imaginary conjugated double points coinciding with the absolute points of the plane. The adopted transformation model which satisfies the conditions for the constructive derivation of these curves, consists of the **2nd** degree series on the **k**, circumference, which is transformed and the pencil of the polar fields of the circumferences in which the transformation takes place and which is determined by the pair of basic circumferences of the $\mathbf{k_1}$ and $\mathbf{k_2}$ pencil.

The adopted transformation model is specialised for obtaining of the real double points by setting the pencil of the circumference polar fields as hyperbolic or parabolic, which have real double elements, double points and double straight lines.

The conditions for the choice and constructive determination of the bicircular curves which have real double points are defined by the relation of the double straight lines of the polar field circumference pencils to the **k** circumference which is transformed in the adopted transformation model.

When the **k** circumference has a pair of common points with any other double straight line, the derived curve has real double point.

The number of possible double points of this curve is 2, because the double straight lines with **k** circumference can have two pairs of the common points at the most.

In the adopted mode one can see the conditions for determination or setting the characteristics of the double points which can appear as the loops or pointed top.

If the series of points on the **k** circumference approaches the double straight line form the different sides, the resulting double point has a form of a loop. When a series of the **k** approaches the double straight line form the same side, the double point has the characteristics of a pointed top.

Bicircular **4th** degree curve will not have real double points with the elliptical pencils of the circumference polar fields or with hyperbolic and parabolic pencils of the polar fields when the **k** circumference has no common points with any of the double straight lines of the pencil.

The selected, and specialised transformation model enables the research and constructive derivation of various 4th degree curve forms from the bicircular curve family.

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KRITERIJUMI ZA KONSTRUKTIVNO ODREDJIVANJE REALNIH DVOSTRUKIH TAČAKA BICIRKULARNIH KRIVIH 4. REDA

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U ovom radu analizirani su kriterijumi pod kojima se mogu odrediti bicirkularne krive 4. reda. Konstruktivno izvodjenje izabranih krivih ostvareno je kvadratnom transformacijom u odgovarajućem modelu transformacije. Podešavanjem performansi modela postignuto je da prikazane bicirkularne krive nemaju realne beskonačno daleke tačke. Analizirani su uslovi pod kojima se, daljom spejalizacijom usvojenog modela transformacije, konstruktivno nalaze realne dvostruke tačke ovih krivih, odredjuje njihov broj i karakteristike: oblik petlje ili oblik šiljka.