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ANALYSIS OF HYPERBOLIC PARABOLOIDS AT SMALL DEFORMATIONS

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Abstract. *A hyperbolic paraboloid, treated from the constructional and mathematical aspects, is analyzed in this paper. In the constructional sense, it is a thin shell of a great bearing capacity and wide usability in spatial structures, either as a complete form or in parts. In the mathematical sense, it is treated as a geometrical surface on which it is possible to determine the rotation field and the field of infinitesimal deformations, and it is rigid.*

1. INTRODUCTION

The contemporary development of building mechanics is oriented towards mathematics, thus offering a wide range of possibilities in the research of usually highly complicated spatial-surface systems. The fact that the fundamental character of mechanics cannot be ignored makes such an approach to the abstract analysis necessary even today. In this, the properties of materials are not neglected at all, and they should be accorded with the building mechanics that is mathematically oriented. The mathematical analysis of these calculations is often complex and not adequate regarding the total time available for the design of structures it is needed for. The approximation, that is the use of computers, reduces long-lasting mathematical calculations and enables constructors to devote more time to the design and construction. Analytical methods are supplemented by the tests on differently sized models, which sometimes simplify significant assumptions of the mathematical methods. The determination of forces and stresses acquired in the models observing and measuring should not be underestimated, yet not overestimated, as well.

This paper represents a review of hyperbolic paraboloid (hereinafter referred to as HP) shells as very frequently used roof structures and points out the possibility of the mathematical analysis application in the case of small deformations.

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2. GEOMETRIC CHARACTERISTICS OF HP

The ruled surface of HP is achieved by the movement of the line AB (ruling line) along the two straight, mutually non-parallel lines AD and BC that do not intersect in the space (Fig. 1).

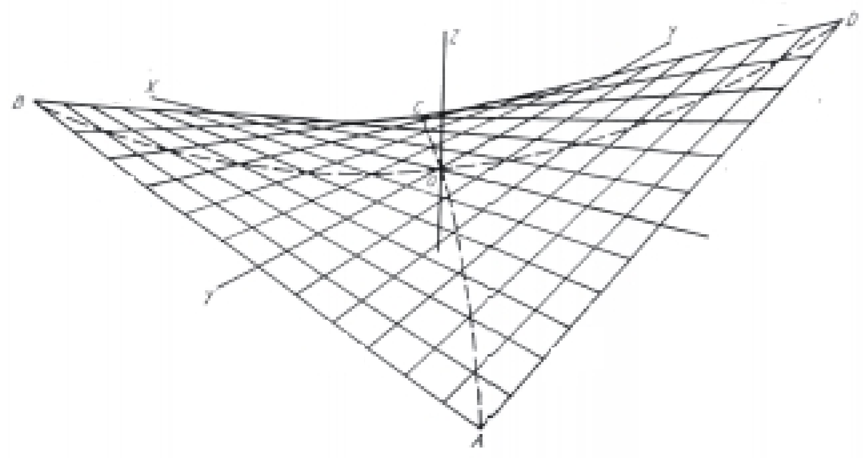

Fig. 1. The HP geometry

A mathematically defined HP is a set of points in the space, whose coordinates, compared to a rectangular coordinate system Oxyz, satisfy the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z .$$

HP originates in the slide of the movable parabola $y^2 = -2b^2z$, $x = 0$, over the immobile parabola $x^2 = 2a^2z$, $y = 0$, so that the parabola axis and plane remain parallel with the starting position.

If we examine the HP level sections, we shall obtain:

- the section of HP and x = h plane is a parabola parallel to y,z-plane

$$y^2 = -2b^2\left(z - \frac{h^2}{2a^2}\right), \quad x = h ,$$

- the section with y, z-plane is obtained for x = 0:

$$y^2 = -2b^2z ,$$

- the section of HP and y = k plane, the plane parallel with y, z-plane, is the parabola

$$x^2 = 2a^2\left(z + \frac{k^2}{2b^2}\right),$$

- especially, the section of HP and x, z-plane (y = 0) is the parabola

$$x^2 = 2a^2z ,$$

- the section with the horizontal plane z = h, h > 0, is the hyperbola

$$\frac{x^2}{2a^2h} - \frac{y^2}{2b^2h} = 1,$$

and with the plane z = h, h < 0, it is

$$-\frac{x^2}{2a^2h} + \frac{y^2}{2ab^2h} = 1,$$

conjugated with the previous hyperbola,

- the section of HP with the plane x, y (z = 0) are two lines

$$y = \pm\frac{b}{a}x.$$

The equation of HP may be written in the following form

$$\frac{x}{a} + \frac{y}{b} = 2t$$
$$\frac{x}{a} - \frac{y}{b} = \frac{z}{t}$$

which shows that HP is a rectilinear surface. The lines in one family do not intersect, while the lines of different families always intersect mutually.

HP may be given in a parametric form as:

$$x = a\rho\cosh v, \quad y = b\rho\sinh v, \quad z = u^2.$$

HP can also be given by the equations:

$$x = a\rho\cos\varphi, \quad y = b\rho\sin\varphi, \quad z = \rho^2\cos 2\varphi$$

or in the form:

$$x = a(u + v), \quad y = b(u - v), \quad z = uv.$$

3. STRUCTURAL DEVELOPMENT OF HP

As spatial roof structures, HPs appeared in 1932 in France. At that time, they were considered extremely daring for the contemporary theoretical approach, research level and building techniques. Although HP is a spatial surface system with double curvature, its realization is relatively simple, as all the roof-boarding elements may be placed in the direction of the straight ruling lines.

Felix Candela, worshiper and author of attractive thin shell structures, particularly of this kind (Fig. 2), greatly contributed to the affirmation of HP.

HP offers unlimited possibilities to architects and constructors in designing and constructing. This warped surface may be applied over any foundation shape (rectangular, triangular, circular, ellipsoid, and so on), and any building which is representative either by its contents or by its position. A harmonious and daring structure of a unique form enables creative expression (Fig. 3).

A HP surface may be continuous and homogenous over the whole foundation span, or it may be assembled of multiplied parts that are copied by plane or axial symmetry. The prefabricated construction with these shells is easily applicable, because the surface, even if it is continuous over the whole foundation span, can be separated into integral prefabricated elements arranged as strips or spatial rectangles that are compounded into a whole girder.

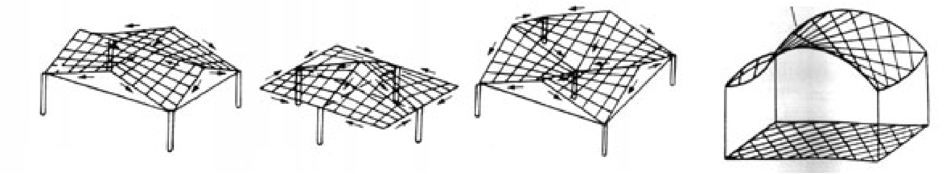

Fig. 2. Hyperbolic paraboloid shapes

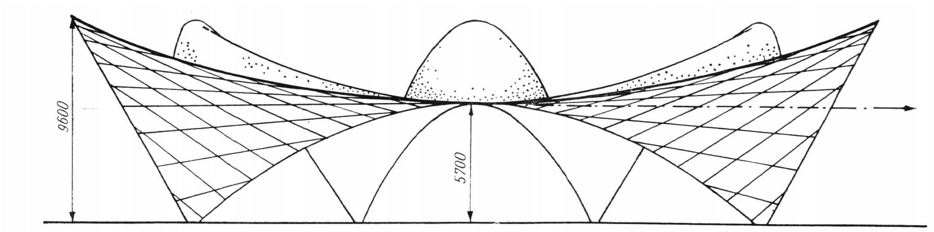

Fig. 3. A combination of several united HPs the roof of a restaurant in Xolchimilco (Mexico)

4. BEARING CAPACITY OF HP

Double-curved surfaces usually have satisfactory bearing capacities, while in HP it is even greater, as the convex curvature stiffens in a way the concave curvature (Fig. 4). Compressive stresses appear along the line *a*, and tensile strains are formed along the line *b*. Having such stress conditions, HP is, righteously, regarded as a membrane, and the calculation of the system bearing capacity is performed according to the membrane theory with small deformations. In HPs, normal forces along the ruling lines have constant values, implying that there is no need for the effect of the shell lateral forces on its edges (edge beams), where constant, longitudinally distributed shearing forces are formed.

The shell principal stresses are created alongside the vertical sections, which make the angle of 45^0 with the ruling lines. Placing edge elements or ribs that accept the compressive or tensile stresses for the designed surface geometry most often solves the reception of shearing forces on the edges. Edge elements may be avoided in highly curved shells, for spans that are shorter than 30 m. In principle, care should

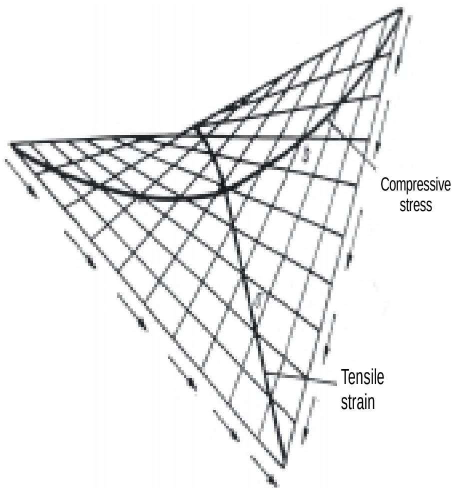

Fig. 4. Membrane stresses in HP

be taken of the edge elements load, that is, their dead weight and its impacts. This is particularly important in cases where they are asymmetrically loaded.

Many possibilities for the calculation of various forms of HP shells are given in the literature. It is necessary to check the bending moments in the HP form that has a negative curve in each point, or in the shells in which the relation of the span and structural height cannot justify the rigid shell concept. Due to the multitude of forms, there are no rules that could be introduced as generally applicable. Numerous theoretical papers that are included in the references [1-8] offer clearly defined and tested recommendations for the calculation of these shell forms. One of such recommendations is that the influence of the bending stress may be neglected in the shells in which the ratio of the height and the length of sides H/a is greater than 0,2 (9). In most cases, HP shells have great safety against bulging due to their curvatures. In shallow shells, the size of elastic and plastic deflection of console parts should be checked.

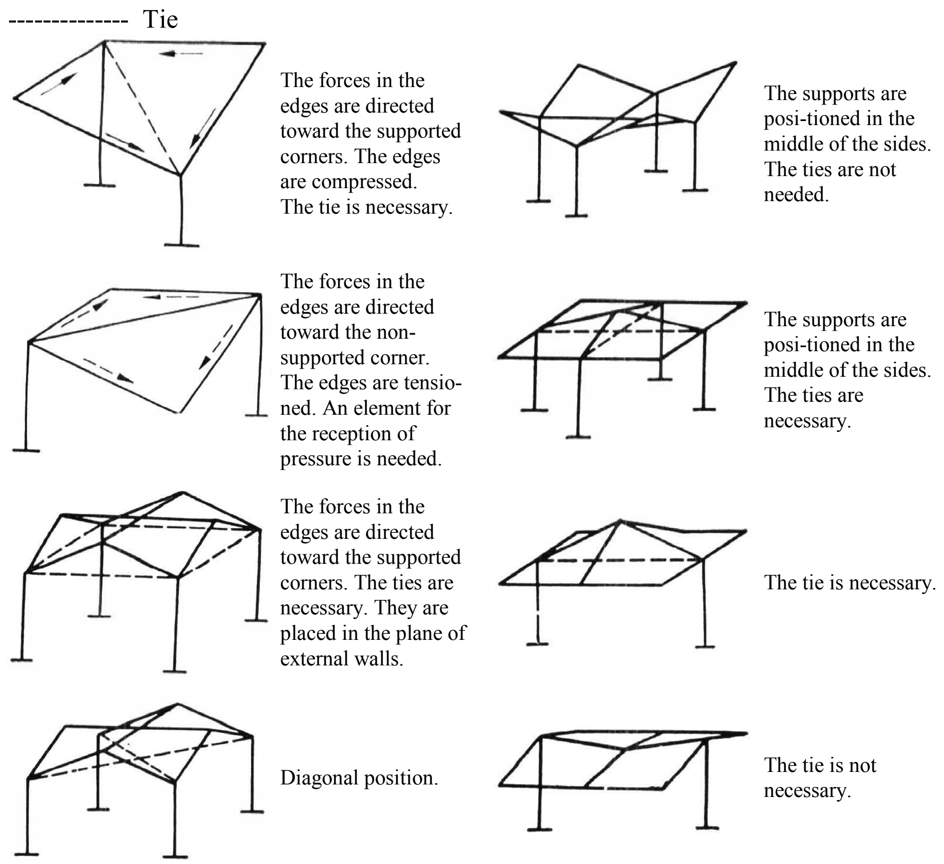

Fig. 5.

The areas around the HP shell edges - edge elements - are the parts with greatest axial forces that are not in balance with the shearing forces. In order to prevent deformation, it is necessary to place diagonal elements (ties or compression members, Fig. 5). The edge elements are in many cases dimensioned not only for the axial forces but also for the bending moments. Satisfactory results are achieved for the prestressing of edge elements.

5. A MATHEMATICAL DEFINITION OF INFINITESIMAL DEFORMATIONS OF SHELLS AND HP

The mathematical approach to the problem of infinitesimal deformations (ID further on) can be presented as a part of the global differential geometry.

Many renowned mathematicians (Cauchy, Liebman, Hielbert, Weil, Blaschke) have dealt with the problem of ID. One of the first works in this field belongs to Cauchy (1813). He proved in it that closed convex polyhedrons are rigid.

In the case that

$$S : \bar{r} = \bar{r}(u, v)\ ,$$

is the vector equation of a regular surface S, and the surface S is included in the family of surfaces S_t ($S = S_0$), expressed by the equation

$$S_t : \bar{r}(u, v, t) = \bar{r}(u, v) + t\bar{z}(u, v)$$

where $t \in R$, $t \to 0$, $\bar{z}$ -continuous differentiable vector function of the C^m ($m \geq 3$) class, defined in the points belonging to the surface S, which is the field of infinitesimal deformations. The surfaces S_t, $t \in R$, $t \to 0$ are the infinitesimal deformations of the surface S if the difference in the linear element squares of these surfaces is an infinitesimal value of a higher order compared to t, $t \to 0$, i.e.

$$ds_t^2 - ds^2 = o(t).$$

This means that the curve arc length variation on the surface is o, $\delta s = 0$, in ID, that is, the arc length of the curve on the surface is stationary in ID. The angles between the curves on the surface are also not changed, as well as other elements that depend on the coefficients of the first fundamental form.

The surface is rigid if it allows only for trivial ID fields. The deformation field is trivial if it has the form of

$$\bar{z} = \bar{a} \times \bar{r} + \bar{b}, \quad \bar{a}, \bar{b} \text{ - constant vectors.}$$

A necessary and sufficient precondition for the surfaces S_t to represent ID of S is that the following is valid:

$$d\bar{r} \cdot d\bar{z} = 0 \tag{1}$$

where $S : \bar{r} = \bar{r}(u, v)$, $\bar{z} = \bar{z}(u, v)$ is the ID field, and – denotes a scalar product, and x denotes vector product.

This equation is equivalent to the three partial equations:

$$\bar{r}_u \cdot \bar{z}_u = 0, \quad \bar{r}_u \cdot \bar{z}_v + \bar{r}_v \cdot \bar{z}_u = 0, \quad \bar{r}_v \cdot \bar{z}_v = 0\ .$$

There is a unique field $\bar{y}(u, v)$ for the ID field $\bar{z}(u, v)$ of the surface, so that:

$$\bar{z}_u = \bar{y} \times \bar{r}_u, \quad \bar{z}_v = \bar{y} \times \bar{r}_v\ ,$$

i.e.

$$d\bar{z} = \bar{y} \times d\bar{r}\,. \tag{2}$$

The rotation field for which the previous relation is valid is the vector field $\bar{y}(u,v)$. As the result of the ID surface, all its elements are subject to the rotation with the rotation vector $\bar{y}(u,v)$.

The field $\bar{s}(u,v)$, determined by the equation

$$\bar{s} = \bar{z} - \bar{y} \times \bar{r}$$

is the field of surface translations at ID with a defined field $\bar{z}(u,v)$.

The derivatives of the vectors $\bar{y}_u$, $\bar{y}_v$ of the rotation field $\bar{y}(u,v)$ are given by the equations

$$\begin{aligned}\bar{y}_u &= \alpha\bar{r}_u + \beta\bar{r}_v \\ \bar{y}_v &= \gamma\bar{r}_u - \alpha\bar{r}_v\end{aligned}$$

where the functions α(u,v), β(u,v), γ(u,v) satisfy the system of partial differential equations:

$$\begin{aligned}&\alpha_v - \gamma_u = \Gamma_{11}^1\gamma - 2\Gamma_{12}^1\alpha - \Gamma_{22}^1\beta \\ &\alpha_u - \beta_v = \Gamma_{11}^2\gamma - 2\Gamma_{12}^2\cdot\alpha - \Gamma_{22}^2\beta \\ &b_{11}\gamma - 2\cdot b_{12}\alpha - b_{22}\beta = 0\end{aligned} \tag{3}$$

where Γ_{jk}^i are Cristoffel's symbols of the surface $\bar{r} = \bar{r}(u,v)$, and b_{ij} are the coefficients of the second fundamental form.

The solution of this system of partial equations determines the functions α, β, γ. The fields $\bar{y}$ and $\bar{z}$ are determined in the following way:

Being that

$$d\bar{y} = \bar{y}_u du + \bar{y}_v dv = (\alpha\bar{r}_u + \beta\bar{r}_v)du + (\gamma\bar{r}_u - \alpha\bar{r}_v)dv\,,$$

is the total differential of the vector function $\bar{y}$, by integrating we get the field $\bar{y}(u,v)$ that is determined in a unilaterally connected surface S. With such determined field $\bar{y}$, the ID $\bar{z}$ field should be further defined. Namely, we have that:

$$d\bar{z} = \bar{y} \times d\bar{r} = (\bar{y} \times \bar{r}_u)du + (\bar{y} \times \bar{r}_v)dv\,. \tag{4}$$

As

$$(\bar{y} \times \bar{r}_u)_v = (\bar{y} \times \bar{r}_v)_u\,,$$

the right side of the equation (4) is the total differential, so the field $\bar{z}(u,v)$ is determined by integration.

We shall examine ID of the surface z = xy, HP. The vector equation of this surface is:

$$\bar{r} = \bar{r}(x,y) = (x, y, xy), \quad (u = x, v = y) \tag{5}$$

or:

$$\bar{r} = x\bar{e}_1 + y\bar{e}_2 + xy\bar{e}_3\,,$$

where $\bar{e}_1, \bar{e}_2, \bar{e}_3$ are mutually perpendicular unit vectors.

As Cristoffel's symbols for this surface are

$$\Gamma^1_{11} = \Gamma^2_{11} = 0, \quad \Gamma^1_{12} = \frac{y}{1+x^2+y^2}, \quad \Gamma^2_{12} = \frac{x}{1+x^2+y^2}, \quad \Gamma^1_{22} = \Gamma^2_{22} = 0,$$

the second fundamental form coefficients are:

$$b_{11} = b_{22} = 0, \quad b_{12} = \frac{1}{\sqrt{a}}, \quad (a = 1 + x^2 + y^2) \ .$$

The equations (3) become:

$$\alpha_y - \gamma_x = -\frac{2y}{a}\alpha, \quad \alpha_x + \beta_y = -\frac{2x}{a}\alpha, \quad -\frac{2}{\sqrt{a}}\alpha = 0,$$

in which

$$\alpha = 0, \quad \beta_y = 0, \quad \gamma_x = 0 \text{, i.e. } \alpha = 0, \quad \beta = \varphi(x), \quad \gamma = \Psi(y) \ ,$$

so, on the basis of this

$$d\bar{y} = \bar{y}_x dx + \bar{y}_y\, dy = \varphi(x)\, \bar{r}_y dx + \Psi(y)\, \bar{r}_x dy = (\Psi(y)dy, \quad \varphi(x)dx, \quad x\varphi(x)dx + y(y)dy)$$

In the case $\bar{y} = (Y_1, Y_2, Y_3)$, the following will be achieved:

$$dY_1 = \Psi(y)dy, \quad dY_2 = \varphi(x)dx, \quad dY_3 = x\varphi(x)dx + y\Psi(y)dy \ .$$

By integrating, we get

$$Y_1 = \int \Psi(y)dy = \lambda(y), \quad Y_2 = \int \varphi(x)dx = \mu(x), \quad Y_3 = \int x\varphi(x)dx + \int y\Psi(y)dy \ .$$

The partial integration results in

$$Y_3 = x\int \varphi(x)dx - \int(\int \varphi(x)dx)dx + y\int \Psi(y)dy - \int(\int \Psi(y)dy)dy,$$
$$Y_3 = x\mu(x) - \int \mu(x)dx + y\lambda(y) - \int \lambda(y)dy \, .$$

The rotation field of HP is

$$\bar{y} = (\lambda(y),\ \mu(x),\ x\mu(x) + y\lambda(y) - \int \mu(x)dx - \int \lambda(y)dy) \ ,$$

in which $\mu(x)$, $\lambda(y)$ are arbitrary functions.

Applying (2), we will determine the ID field. As (5) for the HP is

$$d\bar{r} = (dx, \quad dy, \quad ydx + xdy) \ ,$$

based on (2), it appears that

$$dz = \bar{y} \times d\bar{r} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ \lambda(y) & \mu(x) & x\mu(x) + y\lambda(y) - \int \mu(x)dx - \int \lambda(y)dy \\ dx & dy & ydx + xdy \end{vmatrix} .$$

The HP bending field is obtained by the following integration:

$$\bar{z} = \{y\int\mu(x)dx + \int[\lambda ydy - y\lambda(y)]dy,\ \int[\mu(x)x - \int\mu(x)dx] - x\int\lambda(y)dy,\ -\int\mu(x)dx - \int\lambda(y)dy\}$$

where $\mu(x)$, $\lambda(y)$ are arbitrary functions. It can be proved that the bending field is trivial, i.e. that the HP is a rigid surface with regard to infinitesimal deformations.

6. CONCLUSION

The HP is a ruled surface by which a space based on any desired form can be covered. It is suitable for the systems of roof structures either as a whole roof or as its part. The HP is a thin shell of a great bearing capacity. In spite of its double curvature, its execution is simple due to the possibility of laying the roof boarding in the direction of straight ruling lines. Such structures can also be built by the monolithic assembly of precast units.

The mathematical analysis determined the field of infinitesimal deformations on a HP surface and showed that this type of shell may reasonably be treated as a membrane, i.e. that this surface is rigid.

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ANALIZA HIPERBOLIČNIH PARABOLOIDA PRI MALIM DEFORMACIJAMA

Ljubica Velimirović, Grozdana Radivojević, Dragan Kostić

U radu je analiziran hiperbolički paraboloid tretiran sa konstruktivnog i matematičkog aspekta. U konstruktivnom smislu to je tanka ljuska, velike nosivosti, raznovrsne primene u prostornim konstrukcijama bilo u celini svoje forme ili delovima. U matematičkom smislu tretira se kao geometrijska površ na kojoj je moguće odrediti polje rotacije i polje beskonačno malih deformacija.