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STABILITY PLATE WITH LONGITUDINAL CONSTRUCTIVE DISCONTINUITY

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Abstract. *The influence of longitudinal constructive discontinuity on the stability of the plate in the domain of elastic stability is solved based on the classical thin plate theory. The constructive discontinuities divide the plate into fields of different thickness. The plate has two opposite edges simply supported while the other two edges can take any combination of free, simply supported and clamped conditions. The Levy method is used for the solution of the problem of stability, with the aim of developing an analytical approach when researching the stability of plates with longitudinal constructive discontinuities and also with the aim of obtaining exact solutions for plates with non-uniform thickness. The exact solutions for stability presented herein are very valuable as they may serve as benchmark results for researches in this area.*

Key words: *Levy method, stability criteria, buckling factor, constructive discontinuity.*

1. Introduction

Plates with constructive discontinuity and fields of non-uniform thickness are extensively used in modern structures. By using such plates, it is possible to obtain material saving, weight reduction, stiffness enhancing, designated strengthening, fundamental vibration frequency increasing, etc. A theoretical analysis of the stability of plates with fields of non-uniform thickness is employed in practical engineering designs. Researchers have investigated various forms of thickness variations of the plate that include: a linear function along one direction [1]; a non-linear function along one direction [2] or in both directions [3]. Piecewise constant step functions in one direction are considered in references [4, 5], and in both directions in references [6].

Exact buckling solutions of rectangular plates based on boundary conditions are given in this paper. By using the Levy method, an analytical method for obtaining the solutions is presented. Plates with variable thickness in one direction parallel to the plate edges

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while the thickness is constant in the other direction are considered. The obtained results for the stability of the plate are shown graphically as well as in a tabular form.

The paper is organized in the following manner. Theoretical formulations are presented in Section 2. The Levy method is used to develop the analytical analysis for rectangular plates with constructive discontinuities and fields of non-uniform thickness. Two edges of the plate are simply supported and the other two edges can take any combination of free, simply supported and clamped conditions. In Section 3, the exact solutions for the stability of the rectangular plate when two edges are simply supported and the other two are clamped are presented. The results and discussions are presented in Section 4. This paper ends with a conclusion.

2. Problem Formulation

Figure 1 shows the model of a rectangular plate with $i-1$ ($i = 1,2,...,n$) longitudinal constructive discontinuities which divide the plate into i fields of different thickness h_i. The fields of the plate have a common elastic surface area. The plate is simply supported along two opposite edges that are parallel to the Y-axis, i.e., edges AC and BD. The other two edges CD and AB may be both free or simply supported or clamped. The plate has a constant thickness in the X direction. The origin of the co-ordinate system (X, Y) is set at point C as shown in Figure 1. Assume that a rectangular plate is compressed in its middle plane by forces uniformly distributed along the sides $X = 0$ and $X = a$. The plate is isotropic, elastic with modulus of elasticity E and Poisson's ratio ν. The problem at hand is to determine the critical buckling loads for n rectangular plate fields.

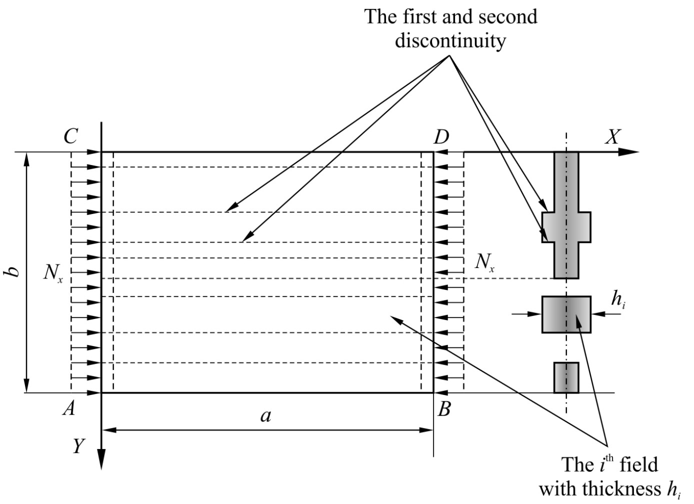

Fig. 1

Based on the classical thin plate theory, the governing differential equation for the i^{th} field is given by [7,8]

$$D_i\left[\frac{\partial^4 w_i}{\partial X^4}+2\frac{\partial^4 w_i}{\partial X^2\partial Y^2}+\frac{\partial^4 w_i}{\partial Y^4}\right]=-N_x\frac{\partial^2 w_i}{\partial X^2}\ ,\ i=1,2\ldots n \tag{1}$$

in which the subscript i refers to the i^{th} field of plate, $w_i(X,Y)$ is the transverse displacement, X and Y are the Cartesian co-ordinates, $D_i=Eh_i^3/(12(1-\nu^2))$ is the flexural rigidity of the field, N_X is the in-plane compressive load.

Using the following transformations:

$$X=ax\ ,\quad 0\le x\le 1\ , \tag{2}$$

$$Y=by\ ,\quad 0\le y\le 1\ , \tag{3}$$

$$\theta=\frac{a}{b}\ , \tag{4}$$

differential equation (1) of the deflection surface can be written in the following form

$$\frac{\partial^4 w_i(x,y)}{\partial y^4}+2\frac{1}{\theta^2}\frac{\partial^4 w_i(x,y)}{\partial x^2\partial y^2}+\frac{1}{\theta^4}\frac{\partial^4 w_i(x,y)}{\partial x^4}+K_i\frac{\pi^2}{\theta^2}\frac{\partial^2 w_i(x,y)}{\partial x^2}=0\ . \tag{5}$$

where K_i is the buckling factor for i^{th} field of the plates and which, for certain materials and load, depends on the dimensions of the plate.

The boundary conditions for the two simply supported edges at x=0 and x=1 are [8]

$$w_i\left(x,y\right)\Big|_{x=0}=0\ ,\qquad w_i\left(x,y\right)\Big|_{x=1}=0 \tag{6}$$

$$\left(M_x\right)_i\Big|_{x=0}=0\ ,\qquad \left(M_x\right)_i\Big|_{x=1}=0 \tag{7}$$

where $(M_x)_i$ is the bending moment defined by

$$\left(M_x\right)_i=D_i\left[\frac{\partial^2 w_i(x,y)}{\partial x^2}+\nu\theta^2\frac{\partial^2 w_i(x,y)}{\partial y^2}\right] \tag{8}$$

The boundary conditions for the other two edges at $y=0$ and $y=1$ are given by

$$\begin{aligned}&w_i(x,y)=0\ \text{ and}\\&(M_y)_i=D_i\left[\frac{\partial^2 w_i(x,y)}{\partial y^2}+\frac{\nu}{\theta^2}\frac{\partial^2 w_i(x,y)}{\partial x^2}\right]=0,\ \text{ if the edge is simply supported}\end{aligned} \tag{9}$$

$$w_i(x,y)=0,\ \frac{\partial w_i(x,y)}{\partial y}=0\ ,\ \text{if the edge is clamped} \tag{10}$$

$$(M_y)_i=D_i\left[\frac{\partial^2 w_i(x,y)}{\partial y^2}+\frac{\nu}{\theta^2}\frac{\partial^2 w_i(x,y)}{\partial x^2}\right]=0\ \text{ and} \tag{11}$$

$$(Q_y)_i=D_i\left[\frac{\partial^3 w_i(x,y)}{\partial y^3}+\frac{(2-\nu)}{\theta^2}\frac{\partial^3 w_i(x,y)}{\partial x^2\partial y}\right]=0\ ,\ \text{if the edge is free.} \tag{12}$$

The subscript i takes the value of either 1 or n, $(M_y)_i$ is the bending moment and $(Q_y)_i$ the transverse force.

By using the Levy approach, the transverse displacement function for the ith field of the plate can be expressed as

$$w_1(x,y) = f_1(y)\sin\alpha_m x\,, \qquad \alpha_m = m\pi\,. \tag{13}$$

where m is the number of sinusoidal half-waves of the buckling in the x direction and $f_i(y)$ is an unknown function to be determined. Eq. (13) satisfies the boundary conditions [eqs. (6) and (7)] for the two simply supported edges at $x=0$ and $x=1$.

In view of Eq. (13), the partial differential equations in Eq. (5) may be reduced to fourth-order ordinary differential equations as

$$f_i^{(4)}(y) - 2\frac{\alpha_m^{\;2}}{\theta^2} f_i''(y) + \left(\frac{\alpha_m^{\;4}}{\theta^4} - K_i\pi^2\frac{\alpha_m^{\;2}}{\theta^2}\right) f_i(y) = 0\,. \tag{14}$$

Depending on the roots of the characteristics equations of the differential equations, there are particular solutions to the above fourth-order differential equations.

The longitudinal constructive discontinuity is at arbitrary distance η from the edge of the plate $y = 0$, $(0 \le \eta \le 1)$.

Along the longitudinal constructive discontinuity between the i^{th} and the $(i + 1)^{\text{th}}$ field of the plate, the following continuity conditions must be satisfied:

$$w_i = w_{i+1}\,,\ \frac{\partial w_i}{\partial y} = \frac{\partial w_{i+1}}{\partial y}\,,\ (M_y)_i = (M_y)_{i+1},\ (Q_y)_i = (Q_y)_{i+1} \tag{15}$$

where w_i and w_{i+1}, $\partial w_i/\partial y$ and $\partial w_{i+1}/\partial y$, $(M_y)_i$ and $(M_y)_{i+1}$, $(Q_y)_i$ and $(Q_y)_{i+1}$ are the deflections, slopes, bending moments and effective shear forse (transversal forse) for the i^{th} and $(i + 1)^{\text{th}}$ field of plate, respectively.

3. Plate With One Discontinuity and Two Clamped Edges

A plate with one longitudinal constructive discontinuity and clamped edges $y = 0$ and $y = 1$ is considered. The discontinuity is at the arbitrary distance $\eta(0\le\eta\le1)$ and it divides the plate into field **1** ($i = 1$) and field **2** ($i = 2$).

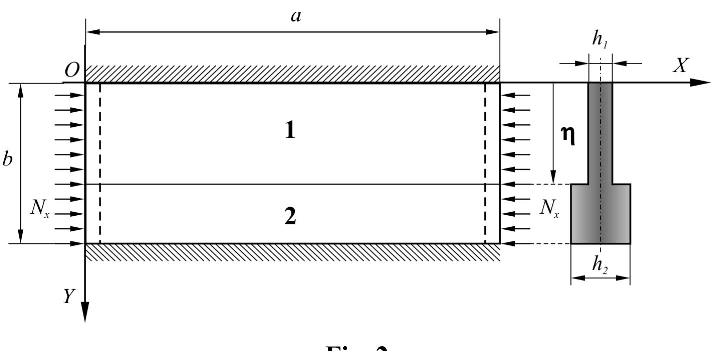

Fig. 2

The flexural rigidities of field **1** and field **2**, and their ratio are:

$$D_1 = \frac{Eh_1^3}{12(1-\nu^2)}, \quad D_2 = \frac{Eh_2^3}{12(1-\nu^2)}, \quad \Psi = \frac{D_1}{D_2} = \left(\frac{h_1}{h_2}\right)^3. \tag{16}$$

The buckling factors are given by the following expressions

$$K_2 = \frac{D_1}{D_2} K_1 = \Psi K \tag{17}$$

Based on expression (2) of the differential equation of the elastic surface of field **1** and field **2,** the plates get the following form

$$\frac{\partial^4 w_1(x,y)}{\partial y^4} + 2\frac{1}{\theta^2}\frac{\partial^4 w_1(x,y)}{\partial x^2 \partial y^2} + \frac{1}{\theta^4}\frac{\partial^4 w_1(x,y)}{\partial x^4} + K\frac{\pi^2}{\theta^2}\frac{\partial^2 w_1(x,y)}{\partial x^2} = 0. \tag{18}$$

$$\frac{\partial^4 w_2(x,y)}{\partial y^4} + \frac{2}{\theta^2}\frac{\partial^4 w_2(x,y)}{\partial x^2 \partial y^2} + \frac{1}{\theta^4}\frac{\partial^4 w_2(x,y)}{\partial x^4} + \Psi K\pi^2 \frac{\partial^2 w_2(x,y)}{\partial x^2} = 0 \tag{19}$$

The solutions of the partial differential equations eqs. (18) and (19) for field **1** and field **2** of the plate in accordance with eq. (13) have the form

$$w_1(x,y) = f_1(y)\sin\alpha_m x, \quad \alpha_m = m\pi. \tag{20}$$

$$w_2(x,y) = f_2(y)\sin\alpha_m x, \quad \alpha_m = m\pi \tag{21}$$

where $f_1(y)$ and $f_2(y)$ are unknown functions to be determined.

Solutions (20) and (21) satisfy the boundary conditions (eqs. (6) and (7)) for the two simply supported edges at $x = 0$ and $x = 1$.

In view of eqs. (20) and (21), the partial differential equations in eqs. (18) and (19) may be reduced to fourth-order type ordinary differential equations as

$$f_1^{(4)}(y) - 2\frac{\alpha_m^2}{\theta^2} f_1''(y) + \left(\frac{\alpha_m^4}{\theta^4} - K\pi^2\frac{\alpha_m^2}{\theta^2}\right) f_1(y) = 0, \tag{22}$$

$$f_2^{(4)}(y) - 2\frac{\alpha_m^2}{\theta^2} f_2''(y) + \left(\frac{\alpha_m^4}{\theta^4} - \Psi K\pi^2\frac{\alpha_m^2}{\theta^2}\right) f_2(y) = 0. \tag{23}$$

The buckling of field **1** and field **2** of the plate in the direction of the y axis is determined by the functions $f_1(y)$ i $f_2(y)$ which are represented in the form

$$f_1(y) = A_1 e^{\lambda y}, \quad f_2(y) = A_2 e^{\lambda' y}. \tag{24}$$

Based on (24) eqs. (22) and (23), the characteristics equation for field **1** is derived

$$\lambda^4 - 2\frac{\alpha_m^2}{\theta^2}\lambda^2 + \left(\frac{\alpha_m^4}{\theta^4} - K\pi^2\frac{\alpha_m^2}{\theta^2}\right) = 0 \tag{25}$$

And the characteristics equation for field **2**

$$(\lambda')^4 - 2\frac{\alpha_m^{\ 2}}{\theta^2}(\lambda')^2 + \left(\frac{\alpha_m^{\ 4}}{\theta^4} - \Psi K\pi^2\frac{\alpha_m^{\ 2}}{\theta^2}\right) = 0\,. \tag{26}$$

The general solution of the differential equation (22) has the form

$$f_1(y) = C_1\cosh\alpha_1 y + C_2\sinh\alpha_1 y + C_3\cos\beta_1 y + C_4\sin\beta_1 y\,, \tag{27}$$

where α_1 and β_1 are the roots of the characteristics equation (25) for the condition $K\pi^2 > \alpha_m^2/\theta^2$

$$\lambda_{1,2} = \pm\sqrt{\frac{\alpha_m^{\ 2}}{\theta^2} + \pi\frac{\alpha_m}{\theta}\sqrt{K}} = \pm\alpha_1\,, \tag{28}$$

$$\lambda_{3,4} = \pm i\sqrt{\pi\frac{\alpha_m}{\theta}\sqrt{K} - \frac{\alpha_m^{\ 2}}{\theta^2}} = \pm i\beta_1\,. \tag{29}$$

The general solution of the differential equation (23) has the form

$$f_2(y) = C_5\cosh\alpha_2 y + C_6\sinh\alpha_2 y + C_7\cos\beta_2 y + C_8\sin\beta_2 y \tag{30}$$

where α_2 and β_2 are the roots of the characteristics equation (26) for the condition $\Psi K\pi^2 > \alpha_m^2/\theta^2$

$$\lambda'_{1,2} = \pm\sqrt{\pi\frac{\alpha_m}{\theta}\sqrt{K\Psi} + \frac{\alpha_m^{\ 2}}{\theta^2}} = \pm\alpha_2\,, \tag{31}$$

$$\lambda'_{3,4} = \pm i\sqrt{\pi\frac{\alpha_m}{\theta}\sqrt{K\Psi} - \frac{\alpha_m^{\ 2}}{\theta^2}} = \pm i\beta_2\,. \tag{32}$$

The constants of integration C_j, $j = 1,2,\ldots 8$ in solutions (27) and (30) are determined from the boundary conditions along the edges $y = 0$ and $y = 1$ and the continuity conditions along the constructive discontinuity.

The boundary conditions on the edges $y = 0$ and $y = 1$ are

$$f_1(y)\big|_{y=0} = 0\,,\ f_1'(y)\big|_{y=0} = 0 \tag{33}$$

$$f_2(y)\big|_{y=1} = 0\,,\ f_2'(y)\big|_{y=1} = 0 \tag{34}$$

In order to ensure the displacement continuities and equilibrium conditions at the discontinuity the following essential and natural conditions (15) must be satisfied

$$f_1(y)\big|_{y=\eta} = f_2(y)\big|_{y=\eta}\,, \qquad f_1'(y)\big|_{y=\eta} = f_2'(y)\big|_{y=\eta} \tag{35}$$

$$\Psi\left[f_1''(y) - \nu\frac{\alpha_m^{\ 2}}{\theta^2}f_1(y)\right]_{y=\eta} = \left[f_2''(y) - \nu\frac{\alpha_m^{\ 2}}{\theta^2}f_2(y)\right]_{y=\eta} \tag{36}$$

$$\Psi\left[f_1''(y) - (2-\nu)\frac{\alpha_m^{\ 2}}{\theta^2}f_1'(y)\right]_{y=\eta} = \left[f_2''(y) - (2-\nu)\frac{\alpha_m^{\ 2}}{\theta^2}f_2'(y)\right]_{y=\eta} \tag{37}$$

From the conditions (33)-(37), a system of homogenous linear algebraic equations in unknown constants of integration is obtained. The buckled form of the equilibrium of the plate becomes possible only if the determinant of this system of equations vanishes. The determinant of the system of equations contains parameters which influence the stability of the plate.

4. Numerical Results and Discussion

The stability criterion is presented in the tables and graphically. Stability region is bellow boundary curves.

Figure 3 shows, for the first half-wave and $\eta = 0.3$, the dependence of the buckling factor K on the plate aspect ratio a/b and the ratio h_2/h_1 of the thickness of the fields of the plate. At the beginning, the values of K decrease with the increase of the ratio a/b, reaching the minimum value and starting from that value, K increases with the increase of the ratio a/b. With the increase of the ratio h_2/h_1, the values of K increase and the minimum values move toward a slightly higher plate aspect ratios a/b.

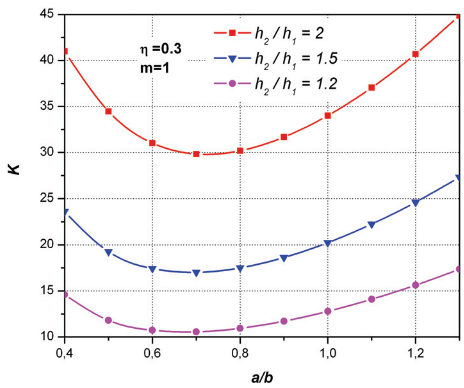

Fig. 3. The buckling factor K in the function of the plate aspect ratio a/b for different a ratio h_2/h_1

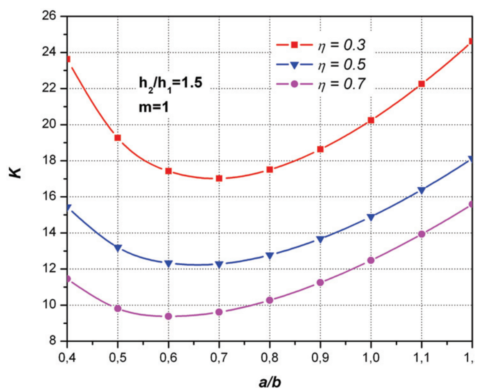

Fig. 4. The buckling factor K in the function of the plate aspect ratio a/b for the different position of the η of the constructive discontinuity

Figure 4 shows, for the first half-wave and the constant ratio h_2/h_1, the dependence of the buckling factor K on the plate aspect ratio a/b of the edges of the plate for different positions of the discontinuity. At the constant ratio h_2/h_1 the value of K decreases with the increase of η because with the increase of η the width of field **1**, which has smaller thickness ($h_1 < h_2$), increases and the width of field **2**, which has greater thickness, decreases. With the decrease of η the minimum values of K move toward a higher ratio a/b.

For the constant position of the discontinuity (η = 0.5), the values of K increase with the increase of the ratio h_2/h_1 (Figure 5).

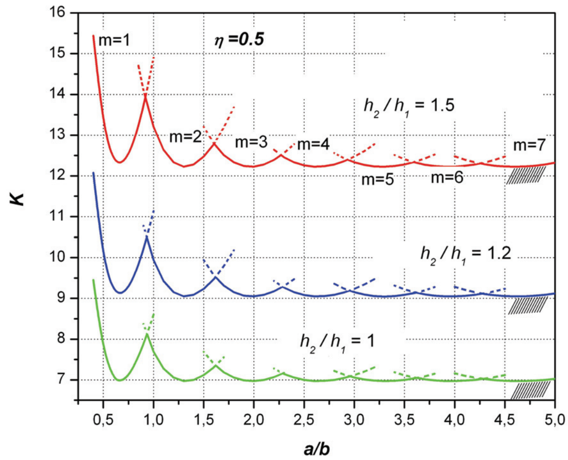

Fig. 5. The buckling factor K as the function of the plate aspect ratio a/b for the different ratio h_2/h_1 and with more half-waves

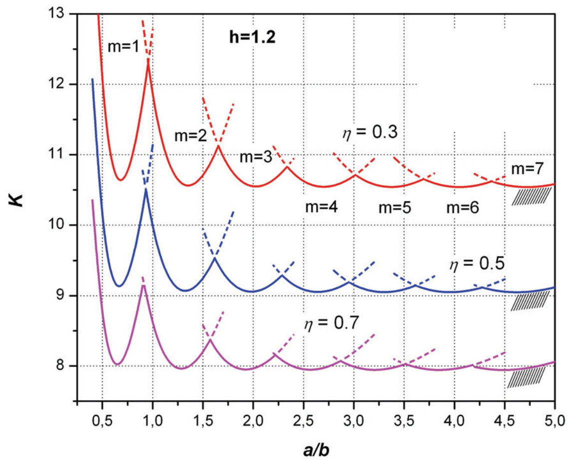

Fig. 6. The buckling factor K as the function of the plate aspect ratio a/b for the different position of η with more half-waves

Figure 6 shows the influence of η on the buckling of the plate in more half-waves. For the constant ratio h_2/h_1 and at a given ratio a/b, the factor K decreases with the increase of η. With the increase of the ratio $a/b > 5$, the values of K approach the minimum values of each individual graph.

For the rectangular plate and the first half-wave (Figure 7), the characteristic point is $h_2/h_1 = 1$ (when there is no discontinuity). With the moving of the position of the discontinuity toward higher values, the value of the buckling factor K decreases because the width of filed **1** is increased. The buckling factor $K = 8.6045$ is for a plate of constant thickness. For the case of $h_2/h_1 < 1$, the value of K increases as η increases because the width of filed **1** increases, which in the given case has greater thickness ($h_1 > h_2$). For a given ratio h_2/h_1 and the case when $h_2/h_1 > 1$, the value of K decreases because the width of filed 1 with smaller thickness ($h_1 < h_2$) increases.

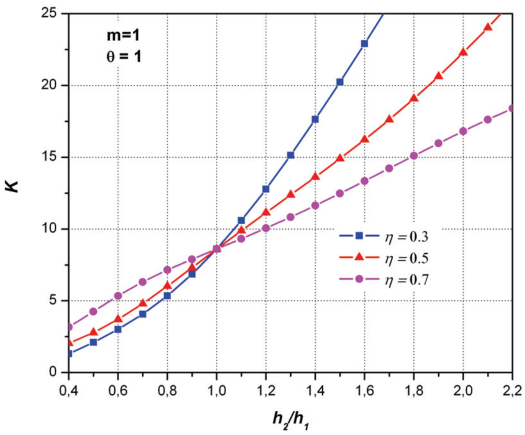

Fig. 7. The buckling factor K in the function of the ratio h_2/h_1 for the different position of η of the constructive discontinuity

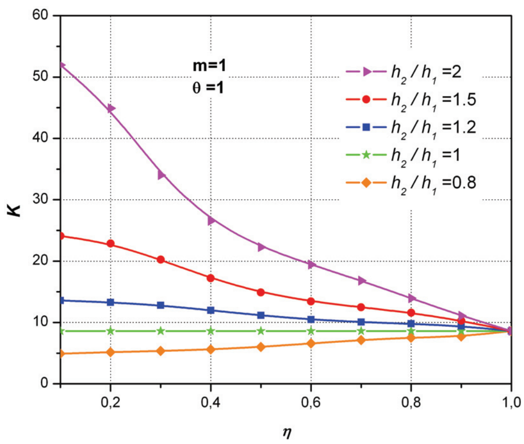

Fig. 8. The buckling factor K in the function of position η of the constructive discontinuity for the different ratio h_2/h_1

Figure 8 shows the dependence of the factor K on the position of the discontinuity for different ratios h_2/h_1 ($m = 1$, $a/b = 1$). With the increase of η, the value of K decreases for the ratio $h_2/h_1 > 1$ and increases for $h_2/h_1 < 1$.

Table 3.4.3.1. Value of the buckling factor K

a/b	h_2/h_1	η=0.3	η=0.5	η=0.7
0.6	1.2	10.7261	9.1515	7.9877
	1.5	17.4265	12.3331	9.38079
	2.0	31.0241	18.3271	11.5142
1	1.0	8.6045	8.6045	8.6045
	1.2	11.8010	9.9392	8.5789
	1.5	19.2690	13.2060	9.8104
	2.0	34.0100	18.8220	11.5260
2	1.2	10.5388	9.0487	7.9581
	1.5	17.0325	12.2272	9.4800
	2.0	30.0290	18.3906	11.9155

The values of the bucling factor K given in the Table are the same as the values obtained in references [5, 9].

5. Conclusion

The stability of a rectangular plate with longitudinal constructive discontinuities has been discussed in the field of elastic stability. Constructive discontinuities divide the plate into fields which are isotropic, vary in thickness and have a common elastic surface area. In its middle plane, the plate is compressed with uniformly distributed forces along two opposing plate edges. The Morris-Levy method has been used to solve the stability problem. Basic theoretical postulates, which are later applied to a particular case, are given. The case when two opposing edges, which are loaded or simply supported, and the other two edges are clamped has been considered. By applying the Levy method of the boundary conditions and the conditions along the constructive discontinuity, a system of equations has been derived. The determinant of the derived system of equations contains parameters which influence the stability of the plate. Based on the obtained solution, numerical analysis has been performed. Stability criteria (the buckling factor) in the function of plate's edges length ratio, plate thickness ratio, the flexural rigidity and the position of the constructive discontinuity are obtained. The minimum values of the critical loads and the boundary curves, below which a stable area is found, are derived. The results presented in this paper provide valuable benchmark solutions for researchers who are developing numerical techniques for buckling analysis of non-uniform thickness plates.

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STABILNOST PLOČE SA PODUŽNIM KONSTRUKTIVNIM DISKONTINUITETOM

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Na osnovu klasične teorije tankih ploča rešavan je uticaj podužnog konstruktivnog diskontinuiteta na stabilnost ploča u oblasti elastične stabilnosti. Konstruktivni diskontinuiteti dele ploču na polja koja imaju različite debljine. Ploča se sastoji od dve naspramne ivice koje su slobodno oslonjene dok preostale dve ivice mogu biti slobodne, slobodno oslonjene ili ukliještene. Za rešenje problema stabilnosti korišćena je Levy-jeva metoda u cilju razvijanja analitičkog pristupa pri proučavanju stabilnosti ploča sa podužnim konstruktivnim diskontinuitetima, odnosno dobijanja tačnih rešenja za ploče sa promenljivom debljinom. Tačna rešenja za stabilnost koja su ovde predstavljena od velikog su značaja jer mogu poslužiti kao referentni rezultati istraživačima u datoj oblasti.

Key words: *Levy-jev metoda, faktor izvijanja, kriterijum stabilnosti, konstruktivni diskontinuitet.*