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FINITE DIFFERENCE METHOD APPLICATION IN DESIGN OF FOUNDATION GIRDER OF VARIABLE CROSS-SECTION LOADED ON ENDS

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Abstract. *Deflection of foundation girder supported by the deformable base has been defined by the system of differential equations, where the differential equation of the elastic line of the girder is of the fourth order. The most convenient solution of the problem is application of numerical procedures, in this case it is the finite difference method. In the paper, the mentioned method is applied in the special case of foundation girder of variable cross section loaded by arbitrary load on its ends.*

Key words: *foundation girder, finite differences method, matrix equation, load*

1. INTRODUCTION

Theory of calculus in the field of foundations and geotechnics in general is related to solution of differential equations describing a physical problem. Their solving through analysis, in a closed form, entails many mathematical difficulties, thus very often various simplifications a re applied in order to obtain acceptable solutions of the problem. When it comes to the difficult tasks, such as design calculus of foundation girders, analytic approach is usually abandoned, giving way for the numerical methods, which can include considerably larger number of influential parameters. The most frequently used numerical methods are the finite difference method and finite element method.

2. PROBLEM DEFINING BY FINITE DIFFERENCE METHOD

The observed foundation girder is loaded by arbitrary external load that varies according to the law $p(x)$ and by the resistance of the base that varies according to the law $q(x)$. The base resistance must be determined in the course of calculation. The girder is longitudinally divided into n equal sections whose length is $c = l / n$ (Figure 1).

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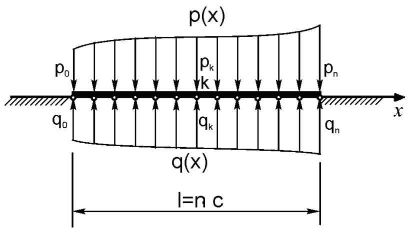

Fig. 1. Discretization of foundation girder

As it is known, the differential equation of the elastic line of a constant cross-section girder can be given in the following form.

$$\frac{d^4 w_x}{dx^4} = \frac{1}{D}(p_x - \ q_x)B \tag{1}$$

where

$D = EI_x$ – is rigidity of the girder to deflection in cross section x (for the foundation girder of constant cross section $D = EI_x$)

w_x – is unknown function of vertical displacement of beam axis point in cross-section x

p_x – function of given external load

q_x – unknown function of reactive load

Basis for solution of this differential equation, through application of finite difference method is substitution of differential by difference calculus. The first derivative is in accordance with that:

$$\frac{dw}{dx} \approx \frac{\Delta\, w}{\Delta\, x} = \frac{1}{2c}(w_{k+1} - w_{k-1}) \tag{2}$$

Further differentiation yields higher derivatives:

$$\frac{d^2 w_k}{dx^2} = \frac{1}{c^2}(w_{k-1} - 2w_k + w_{k+1})$$

$$\frac{d^3 w_k}{dx^3} = \frac{1}{2c^3}(-w_{k-2} + 2w_{k-1} - 2w_{k+1} + w_{k+2})$$

$$\frac{d^4 w_k}{dx^3} = \frac{1}{c^4}(w_{k-2} - 4w_{k-1} + 6w_k - 4w_{k+1} + w_{k+2}) \tag{3}$$

Introducing the last equation from (3) into equation (1) an expression for determination of the ordinate of elastic line of the girder in an arbitrary point k is obtained k:

$$\frac{w_{k-2} - 4w_{k-1} + 6w_k - 4w_{k+1} + w_{k+2}}{c^4} = \frac{1}{D}(p_k - q_k)B \qquad (k = 0,1,........,n) \tag{4}$$

Equation (4) can be written for each dividing point of the girder, so that a system of $n + 1$ equation is obtained, defining the elastic line of the girder. In practical terms, the differential equation of the elastic line of the girder (1) is reduced to the system of linear algebraic equations (4). This further implies that equations for points *0, 1, n-1 and n* contain also the ordinates of the elastic line of points which are outside the girder:

For point 0 $$w_{-2} - 4w_{-1} + 6w_0 - 4w_1 + w_2 = \frac{c^4}{D}(p_0 - q_0)B$$

For point 1 $$w_{-1} - 4w_0 + 6w_1 - 4w_2 + w_3 = \frac{c^4}{D}(p_1 - q_1)B$$

For point $(n-1)$ $$w_{n-3} - 4w_{n-2} + 6w_{n-1} - 4w_n + w_{n+1} = \frac{c^4}{D}(p_{n-1} - q_{n-1})B$$

For point n $$w_{n-2} - 4w_{n-1} + 6w_n - 4w_{n-1} + w_{n+2} = \frac{c^4}{D}(p_n - q_n)B \quad (5)$$

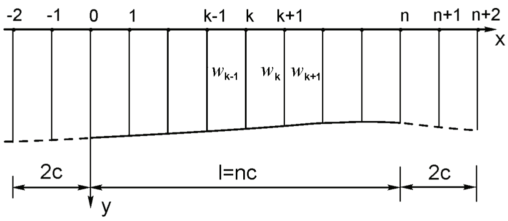

Fig. 2. Ordinate of the elastic line of points on the girder and outside

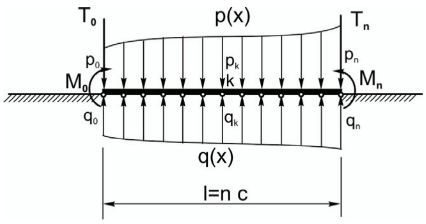

Fig. 3. Foundation girder with ends loaded by moments and transversal forces

In a system of $n + 1$ equation (4), there occur $n + 5$ of unknown ordinates of the elastic line. The ordinates in the points outside the girder: w_{-2}, w_{-1}, w_{n+1}, w_{n+2} – are excessive and they are eliminated using the contour conditions. For elimination of the excess of four unknowns, in agreement with [2], contour conditions are used in which the moments and transversal forces on the ends of the girder are equal to null. Then, the contour conditions agreeing with the terms (3) can be written in the following form:

$x = 0$ $$M_0 = 0 \;\Rightarrow\; \frac{d^2 w_0}{dx^2} = \frac{w_{-1} - 2w_0 + w_{+1}}{c^2} = 0$$

$$T_0 = 0 \;\Rightarrow\; \frac{d^3 w_0}{dx^3} = -\frac{w_{-2} - 2w_{-1} + 2w_{+1} - w_{+2}}{2c^3} = 0$$

$x = n$ $$M_n = 0 \;\Rightarrow\; \frac{d^2 w_n}{dx^2} = \frac{w_{n-1} - 2w_n + w_{n+1}}{c^2} = 0$$

$$T_n = 0 \;\Rightarrow\; \frac{d^3 w_n}{dx^3} = -\frac{w_{n-2} - 2w_{n-1} + 2w_{n+1} - w_{n+2}}{2c^3} = 0 \quad (6)$$

The unknown ordinates w_{-2}, w_{-1}, w_{n+1}, w_{n+2} in the points outside the girders, from the system of equations (6) are determined in function of the ordinates: w_0, w_1, w_2, w_{n-2}, w_{n-1}, w_n in the points which lie on the girder. When the values calculated in this way are introduced in the system of form equations (4), written for each dividing point of the girder, a system of linear algebraic equations is obtained, which can be represented in the matrix form for the sake of convenience

$$[A]\{w\} = \frac{c^4}{D}(\{p\} - \{q\})B \tag{7}$$

where:

$[A]$ – is the square matrix of n+1 order whose elements are coefficients with unknown w

$w = [w_0, w_1, ..., w_n]$ – unknown ordinate column vector of elastic girder line points $0,1,.......,n$ $w = [w_0, w_1, , w_n]$

$\{p\} = \{p_0, p_1, ..., p_n\}$ – external girder load ordinate column vector in dividing girder points $0,1,.......,n$

$\{q\} = \{q_0, q_1, ..., q_n\}$ – unknown ordinate column vector of base resistance in dividing girder points $0,1,.......,n$

However, conditions on the ends of the girder can be differently defined. When the moments and transversal forces at the ends of girder are other than zero, (Figure 3), the contour conditions in these points assume the following form:

For the point $x = 0$ $M = M_0$ and $T = T_0$

$$\frac{d^2 w_0}{dx^2} = \frac{w_{-1} - 2w_0 + w_1}{c^2} = -\frac{M}{D}$$

$$\frac{d^3 w_0}{dx^3} = -\frac{w_{-2} - 2w_{-1} + 2w_1 - w_2}{2c^3} = -\frac{T_0}{D}$$

for the point $x = l$ $M = M_n$ and $T = T_n$

$$\frac{d^2 w_n}{dx^2} = \frac{w_{n-1} - 2w_n + w_{n+1}}{c^2} = \frac{M_n}{D}$$

$$\frac{d^3 w_n}{dx^3} = -\frac{w_{n-2} - 2w_{n-1} + 2w_{n+1} - w_{n+2}}{2c^3} = -\frac{T_n}{D_n} \tag{8}$$

In this system of four equations, the ordinates w_{-2}, w_{-1}, w_{n+1}, w_{n+2} are unknown. By solving the equation system, the mentioned ordinates assume the following values:

$$w_{-1} = -\frac{M_0 c^2}{D} + 2w_0 - w_1$$

$$w_{-2} = -2\frac{M_0 c^2}{D} + 2\frac{T_0 c^3}{D} + 4w_0 - 4w_1 + w_2$$

$$w_{n+1} = -\frac{M_n c^2}{D} - w_{n-1} + 2w_n$$

$$w_{n+2} = -2\frac{M_n c^2}{D} - 2\frac{T_n c^3}{D} + w_{n-2} - 4w_{n-1} + 4w_n \tag{9}$$

After introducing the values calculated in such a manner, into the equation system (4) it is transformed into:

$$2w_0 - 4w_1 + 2w_2 = \frac{c^4}{D}(p_0^{'} - q_0)B$$

$$-2w_0 - 5w_1 - 4w_2 + w_3 = \frac{c^4}{D}(p_1^{'} - q_1)B$$

$$w_{k-2} - 4w_{k-1} + 6w_k - 4w_{k-1} + w_{k+2} = \frac{1}{D}(p_k^{'} - q_k)B$$

$$w_{n-3} - 4w_{n-2} + 5w_{n-1} - 2w_n = \frac{c^4}{D}(p_{n-1}^{'} - q_{n-1})B$$

$$2w_{n-2} - 4w_{n-1} + 2w_n = \frac{c^4}{D}(p_n^{'} - q_n)B \quad (10)$$

Where $p'_0, p'_1, p'_{n-1}\ p'_n$ are modified ordinates of external load in dividing girder points 0, 1, n–1 and n respectively. Their values are determined according to the terms obtained after the equation systems is arranged (10):

$$p_0^{'} = p_0 - 2\frac{M_0}{Bc^2} - 2\frac{T_0}{Bc}$$

$$p_1^{'} = p_1 + \frac{M_0}{Bc^2}$$

$$p_{n-1}^{'} = p_{n-1} + \frac{M_n}{Bc^2}$$

$$p_n^{'} = p_n - 2\frac{M_n}{Bc^2} + 2\frac{T_n}{Bc} \quad (11)$$

In this way, $\{p\}$ vector is modified into $\{p'\}$ vector. Accordingly, vector of given external load $\{p'\}$, in the case when the girder ends are loaded by moments or transversal forces will be different from the $\{p\}$ vector in cases when the ends are free of load. Only by the ordinates of external load in the points 0, 1, n-1 and n. In this way, the matrix equation (7) is transformed into.

$$[A]\cdot\{w\} = \frac{c^4}{D}(\{p^{'}\} - \{q\})B \quad (12)$$

that is,

$$\frac{D}{Bc^4}[A]\cdot\{w\} = (\{p^{'}\} - \{q\})B \quad (13)$$

When the beam has a variable cross-section, the matrix equation (13) is transformed in a following way:

$$\frac{[D']}{c^4}[A]\cdot\{w\} = \{p^{'}\} - \{q\} \quad (14)$$

where

$[D']$ – is the square matrix of $n+1$ order, whose elements are relations of beam rigidity at the individual segments ($D_k = EI_k$) and beam width at the same segments (B_k), that is:

$$[D^{'}]=\begin{bmatrix} \frac{D_0}{B_0} & 0 & 0 & & & \\ 0 & \frac{D_1}{B_1} & 0 & & & \\ & . & . & . & . & \\ & & & 0 & 0 & \frac{D_n}{B_n} \end{bmatrix} \tag{15}$$

Final solution of matrix equations of (7), (12) or (14) forms, requires previous defining of relations between vectors $\{w\}$ and $\{q\}$.

3. CONCLUSION

By the described procedure, implementing finite difference method, the differential equation of the beam elastic line is reduced to a matrix equation (14). It is more general in type in respect to the equation (7) which is derived for the constant cross-section girder whose ends are not loaded. According to that, the matrix equation of form (14) will be applied for the girders of variable cross-section and for the girders whose ends are loaded by moments and transversal forces. The matrix form is very convenient for mathematical representation and creating of computer program intended for solution of the presented problem.

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REFERENCES

1. Bonić Z.: "Prilog teoriji proračuna temeljnih nosača na elastičnoj podlozi", magistarski rad, Građevinsko-arhitektonski fakultet u Nišu, Niš 2000, p. 7-26
2. Desai S.C., Christian J. T.: "Numerical methods in geotechnical engineering", McGraw-Hill Book Company, United States, 1977 [3]. Nonveiller E. Mehanika tla i temeljenje građevina, Školska knjiga, Zagreb, 1991, pp. 545-554
3. Prolović V., BONIĆ Z.: The calculation of foundation girders in equivalent elastic semispace, Facta Universitatis, 1999, Vol. 2, NO1, 1999, pp. 61-67
4. Prokić A., Lukić D.: Dynamic behaviorof braced thin walled beams, International Applied Mechanics, 2007, Vol.43, pp. 1290-1303,
5. Scott F. R.: "Foundation analysis", Prentide-Hall, Inc., Englewood Cliffs, United States, 1981.
6. Stevanović S.: "Fundiranje I", Naučna knjiga, Beograd 1989, pp. 275-291
7. Tomlinson M.J.: "Foundation design and construction", Pearson Prentice Hall, Edinburgh, 2001.

PRIMENA METODE KONAČNIH RAZLIKA U PRORAČUNU TEMELJNOG NOSAČA PROMENLJIVOG POPREČNOG PRESEKA OPTEREĆENOG NA KRAJEVIMA

Verka Prolović, Zoran Bonić

Savijanje temeljnog nosača koji se oslanja na deformabilnu podlogu definisano je sistemom linearnih diferencijalnih jednačina, pri čemu je diferencijalna jednačina elastične linije nosača, četvrtog reda. Za rešavanje postavljenog problema je primena numeričkih postupaka, od kojih je ovde usvojena metoda konačnih razlika. U radu je prikazana primena navedene metode za specijalni slučaj temeljnog nosača promenljivog poprečnog preseka opterećenog proizvoljnim opterećenjem i na njegovim krajevima.