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MINIMAL SURFACES FOR ARCHITECTURAL CONSTRUCTIONS

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Abstract. *Minimal surfaces are the surfaces of the smallest area spanned by a given boundary. The equivalent is the definition that it is the surface of vanishing mean curvature. Minimal surface theory is rapidly developed at recent time. Many new examples are constructed and old altered. Minimal area property makes this surface suitable for application in architecture. The main reasons for application are: weight and amount of material are reduced on minimum. Famous architects like Otto Frei created this new trend in architecture. In recent years it becomes possible to enlarge the family of minimal surfaces by constructing new surfaces.*

Key words: *Minimal surface, spatial roof surface, area, soap film, architecture*

1. INTRODUCTION

If we dip a metal wire-closed space curve into a soap solution, when we pull it out, a soap film forms. A nature solves a mathematical question of finding a surface of the least surface area for a given boundary. Among all possible surfaces soap film finds one with the least surface area.

Deep mathematical problems lie in the theory behind.

The theory of minimal surfaces is a branch of mathematics that has been intensively developed, particularly recently. On the base of this theory we can investigate membranes in living cells, capillary phenomena, polymer chemistry, crystallography. Minimal surfaces are also applied in architecture.

In spite of the fact that it seems that soap film easily solves mathematical problem of finding minimal surface for the boundary curve, attempts to solve some basic problems as well as to give description of minimal surfaces was hard work in mathematics for over 200 years.

The main fields of mathematics contributing to minimal surface theory are differential geometry, complex analysis, theory of partial differential equations and calculus of variations.

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In the recent time, as in many other areas, great progress was made by using computers. This new technology enabled researchers to enlarge the family of minimal surfaces as well as to confirm old ideas, to see old abstract known minimal surfaces, to alter them and to check their properties.

Theoretical investigation of these surfaces is useful for application of this knowledge in further investigation of forms in architecture.

One of the first uses of computers was for the analysis of structures, using theories that have been developed continuously from the 16th century.

Minimal surfaces are extremely stable as physical objects, and this can be an advantage in many kinds of structures. From architects' point of view computerized illustrations of some of minimal surfaces are intrigued by the possibility of adapting them to structures, both interior and exterior.

1. INFINITESIMAL DEFORMATION OF A SURFACE WITH A FIXED CONTOUR

Minimal surfaces are defined as surfaces of the smallest area spanned by a given space curve.

The Plateau's problem is the problem in calculus of variations to find the minimal surface for a boundary with specified constraints (having no singularities on the surface).

In 1873 a physicist named Joseph Plateau observed that soap film bounded by wire appeared to form minimal surfaces. The problem named after him, Belgian physicist experimentally solved for some special cases. Jess Douglas 1931 solved this problem. In general, there may be one, multiple, or no minimal surfaces spanning a given closed curve in a space.

Soap film must go to the state at which the surface area is minimized in order to minimize surface tension and reach equilibrium.

Let us consider infinitesimal deformation of a surface $S : \vec{r} = \vec{r}(u,v), (u,v) \in D \subset R^2$, including this initial surface in a family of surfaces

$$S_t : \vec{r}_t(u,v) = \vec{r}(u,v) + t\vec{z}(u,v), t \to 0, t = t(u,v), S_0 = S, \tag{1}$$

where deformation field is surface normal. Differentiating with respect to u and v, we obtain

$$\vec{r}_{tu}(u,v) = \vec{r}_u(u,v) + t_u\vec{z}(u,v) + t\vec{z}_u(u,v),$$
$$\vec{r}_{tv}(u,v) = \vec{r}_v(u,v) + t_v\vec{z}(u,v) + t\vec{z}_v(u,v).$$

From here, neglecting terms of higher order then the first, we have

$$E_t = E - 2te + o(t^2),$$
$$F_t = F - 2tf + o(t^2),$$
$$G_t = G - 2tg + o(t^2).$$

Introducing the mean curvature

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}, \text{we get}$$

$$E_t G_t - F_t^2 = (EG - F^2)(1 - 4tH) + o(t^2)\text{, and}$$

$$\sqrt{E_t G_t - F_t^2} = \sqrt{(EG - F^2)}(1 - 2tH) + o(t^2)\,.$$

Let S_t (1) be a regular surface in R^3, then the area enclosed by a fixed contour,

$$A(t) = \iint \sqrt{E_t G_t - F^2{}_t}\, dudv, \tag{1}$$

and

$$A(0) = \iint \sqrt{EG - F^2}\, dudv, \tag{2}$$

area on the surface S enclosed by a same fixed contour. The first variation

$$\begin{aligned} A'(0) &= \lim_{t\to 0} \frac{A(t) - A(0)}{t} = \lim_{t\to 0} \iint \frac{\sqrt{EG - F^2}(1 - 2tH) - \sqrt{EG - F^2} + o(t^2)}{t} dudv \\ &= -2\iint H\sqrt{EG - F^2}\, dudv = -2\iint HdA. \end{aligned} \tag{3}$$

In the case when the mean curvature vanishes H=0, we have minimal surface i.e. the surface of minimum area passing through a closed curve.

A Monge patch

$$\vec{r} = \vec{r}(u, v, h(u, v)), \tag{4}$$

is a minimal surface if

$$(1 + h^2{}_v)h_{uu} - 2h_u h_v h_{uv} + (1 + h^2{}_u)h_{vv} = 0, \tag{5}$$

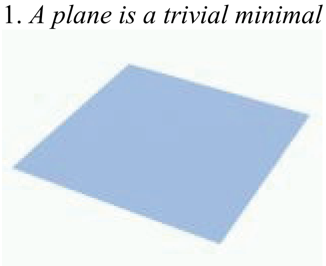
which is Lagrange's equation of the minimal surface. It follows as an immediate consequence of the fact that for a Monge equation (4) mean curvature is

$$H = \frac{(1 + h^2{}_v)h_{uu} - 2h_u h_v h_{uv} + (1 + h^2{}_u)h_{vv}}{(1 + h^2{}_u + h^2{}_v)^{\frac{3}{2}}} \text{ and } H = 0.$$

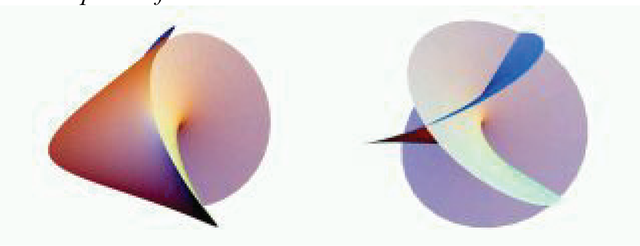
2. EXAMPLES OF MINIMAL SURFACES

We will here note some of the types of minimal surfaces suitable for application at civil engineering and architecture. Pictures are made using program package *Mathematica*.

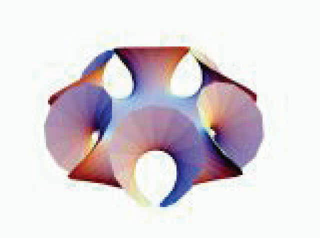
1. *A plane is a trivial minimal surface*:



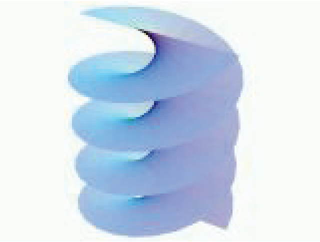
2. *Enneper surface*:



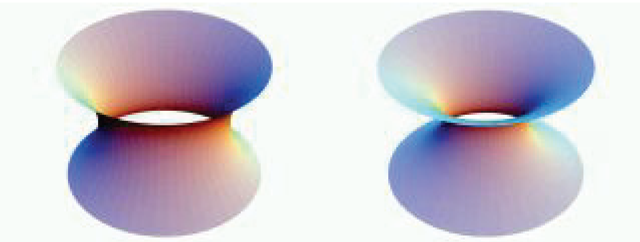
3. *Higher order f Enneper surfaces*:



4. *The helicoid*:

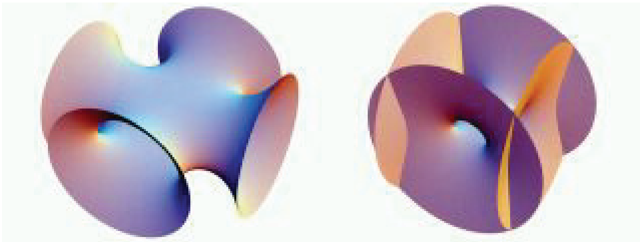


5. *The catenoid*:

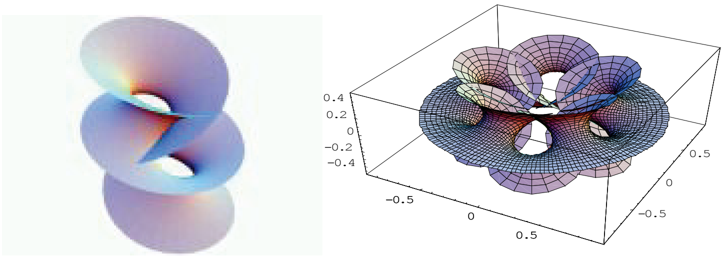


1776. Meusnier found catenoid and helicoid. Helicoid and catenoid are the only two ruled minimal surfaces. Catenoid can isometricaly be bent to helicoid through isometrical minimal surfaces

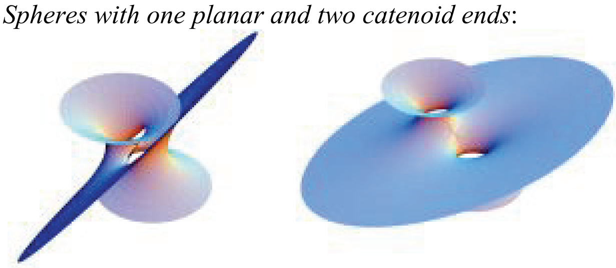
6. *Jorge-Meeks surfaces with n ends:*
n=4



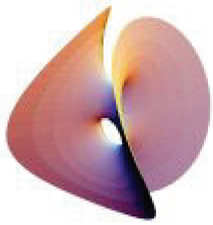
7. *Richmond surface*:



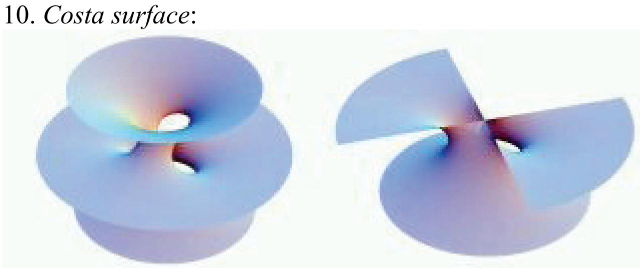
8. *Spheres with one planar and two catenoid ends*:



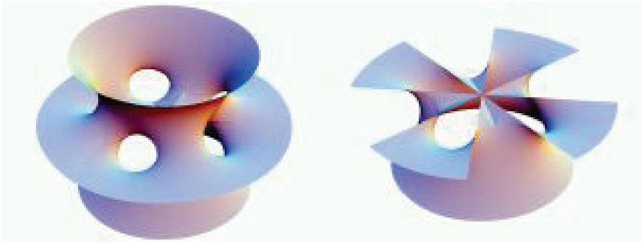
9. *Chen-Gackstatter surface*:



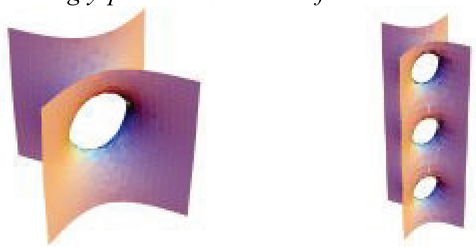
10. *Costa surface*:



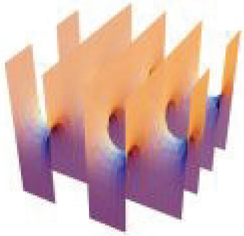
11. *Costa-Hoffman-Meeks surfaces of genus k*:
k=3



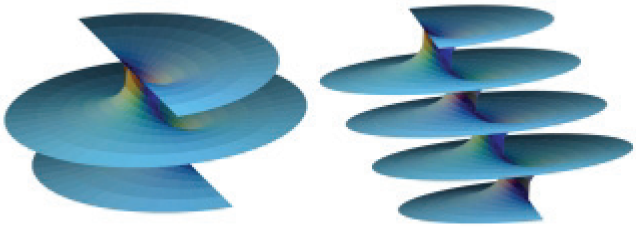
12. *The singly-periodic Scherk surface*:



13. *The doubly-periodic Scherk surface*:



14. *The singly-periodic Riemann's staircase*:



3. APPLICATION OF MINIMAL SURFACES IN ARCHITECTURE

Increasing number of designers and architects are aware of the fact that knowledge of form is a very important aspect of design of structures.

The main aim of this work is to point out to a class of surfaces that are suitable for application in architecture.

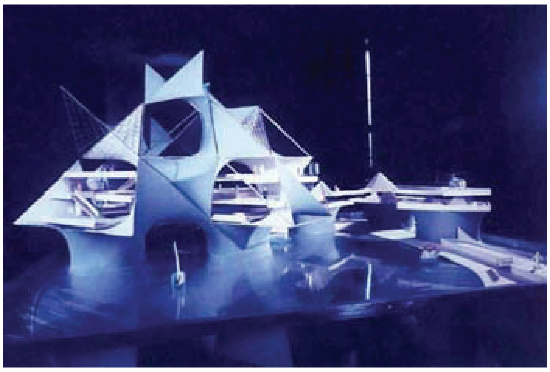
The main reason for application of minimal surfaces in architecture lies in the definition. Having the least area property minimal surface is used for light roof constructions, form-finding models for tents, nets and air halls. Among the surfaces having the same boundary minimal surface is the surface of the least area. It's weight is therefore less and the amount of material is reduced on minimum.

Form of huge soap films are spanned by the boundary and fixed at some points. Balanced surface tension stabilizes the whole construction since the tension is in equilibrium at each point on the roof, as on a soap film.

Famous architects and among them Otto Frei created minimal roofs. German Pavillon for Expo 1968 at Montreal is one of them. The Munich Olimpic Stadium and Kongreshall in Berlin are the others. Otto Frei was also experimenting with hanging chain nets and soap films.

Hyperbolic paraboloid is a ruled surface. Sometimes it is mentioned to be a minimal surface, but it is not. The only ruled surfaces among minimal surfaces are catenoid and helicoid, and plane. However hyperbolic paraboloid at some conditions can be used as good and simple approximation of minimal surface.

15. *Experimental building by architect Michael Burt*



The architect Michael Burt, called the 'Hexahyp', at the Israel Institute of Technology, Haifa, Israel, and the picture he made on parts of its surface. The fiberglass covering consists of saddle-back shaped surfaces of the type that would be assumed by soap films stretched between outlines of the supporting structure. These surfaces, called minimal surfaces, provide the maximum of strength for the minimum amount of material.

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MINIMALNE POVRŠI U ARHITEKTONSKIM KONSTRUKCIJAMA

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Minimalne površi su površi najmanje površine za datu granicu. Ekvivalentna je definicija da su to površi sa nultom srednjom krivinom. Teorija minimalnih površi se rapidno razvija u novije vreme .Konstruišu se mnogi novi primeri. Osobina da su to površi sa minimalnom površinom čini ih pogodnim za primenu u arhitekturi. Glavni razlog za to je da su težina i količina materijala svedeni na minimum. Poznati arhitekte kao Otto Frei su kreirali novi trend u arhitekturi. U poslednje vreme je postalo moguće uvećati familiju površi koje se primenjuju konstrukcijom novih.