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LIMIT ANALYSIS OF BEAMS UNDER COMBINED STRESSES

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Abstract. *The problem of the determination of limit bearing capacity of beam cross section under pure bending, eccentric tension, pure shear, as well as combined stress is considered in this paper. The influence functions of the bending moment and axial force, as well as the bending moment, axial and shear force on the cross section limit bearing capacity in case of rectangular and I beam cross section are derived.*

Key words: *Limit Analysis, Beams, Stress*

1. INTRODUCTION

This paper discusses the issue of the ultimate bearing capacity of a beam cross section. The analysis concerns the limit loading capacity of a cross section under pure bending, eccentric stress and shear, as well as combined shear and bending. The following assumptions are, thus, taken into consideration:

a) material is perfectly elastic – plastic without hardening;
b) material is homogeneous and isotropic;
c) the plane cross sections remain plane and orthogonal to the deformed material axis – Bernoulli`s hypothesis.

It is also assumed that the entire cross sections possesses at least one symmetry axis.

2. PURE BENDING OF BEAMS

In the case when a beam is stressed with pure bending, bending moment at cross section will be determined as per equation:

$$M = \int_A \sigma\, y\, dA. \tag{1}$$

Yield hinge or "plastic hinge" is defined as such a cross section in any structure where all normal bending stresses (σ) along the total height of the section reached the limit of great

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extension – yield stress R_e. The appropriate bending moment is, thus, the greatest one that a section can support. At this point each fiber in the beam is yielding in either tension or compression. Corresponding bending moment is known as limit moment, yield moment or full plastic moment capacity, or simpler, "plasticity moment". It is determined by the following equation:

$$M_p = \int_A R_e \, y \, dA = R_e \, Z, \tag{2}$$

where Z, called the plastic modulus, is equal to the sum of the static moments of the parts of the cross section above and below neutral axis with respect to the neutral axis at the stage of full plastification of cross section.

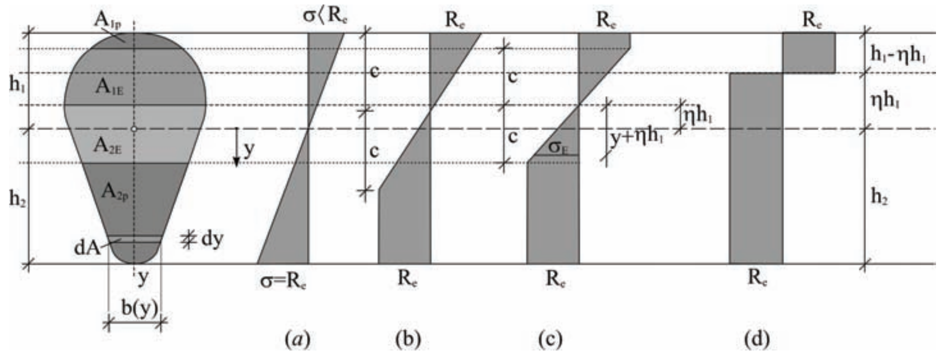

Fig. 1. Stress distribution at cross section with one symmetry axis sample:
(*a*) elastic, (b) elastic – plastic, (c) elastic – plastic, (d) maximum plastic $M=M_p$

When analyzing the arbitrary cross section with one symmetry axis at various phases of loading, thus paying special attention to elastic – plastic and full plastic range, the general expressions for corresponding bending moment can be derived, which can later be applied to any cross section containing one or two symmetry axes.

The maximum elastic bending moment will occur when $\sigma = R_e$ at the most distant fiber of the beam (Fig. 1. (*a*)). Corresponding moment does not represent the carrying capacity of the beam. If the moment is still further increased, the outermost fibers will be stressed to the yield stress. The central fibers, however, will still be able to carry more load. The stage presented at Fig. 1. (c) shows that the area of elasticity is being spread up to the fibers that are at "c" position in respect to the temporary position of the neutral axis and this position is being defined with the changing ηh_1 distance. At that moment, the bending moment is:

$$M = \int_{A_{1E}} \sigma_E \, y \, dA + \int_{A_{2E}} \sigma_E \, y \, dA + \int_{A_{1p}} R_e \, y \, dA + \int_{A_{2p}} R_e \, y \, dA, \tag{3}$$

where:

- σ_E is elastic stress in the part of the cross section which is not plastified jet and which can be determined with the use of the proportion $\sigma_E = \frac{y}{c} R_e$;
- y is fiber distance from the main central axis of bending;

- A_{1E}, A_{2E} stand for the areas of cross section elastic parts that are above and below the neutral axis;
- A_{1p}, A_{2p} areas of yielded parts of cross section,

or:

$$M = R_e\left[\frac{1}{c}\left(\int_0^c y^2 b(y)dy + \int_0^c y^2 b(y)dy\right) + \left(-\int_{-h_1}^{-(c+\eta h_1)} y b(y)dy + \int_{c-\eta h_1}^{h_2} y b(y)dy\right)\right]. \quad (4)$$

At the moment of full plastification of the cross section, the elastic part of the cross section disappears, distance "c" gains zero value (Fig. 1. (d)) and the appropriate bending moment is fully plastic moment:

$$M_p = R_e Z = R_e\left(-\int_{-h_1}^{-\eta h_1} y b(y)dy + \int_{c-\eta h_1}^{h_2} y b(y)dy\right). \quad (5)$$

If the case is the one of a rectangular cross section with the dimensions b x 2h (Fig. 2.), the equation (3) then looks as follows:

$$M = R_e\left[\frac{2}{c}\int_{A_{1E}} y^2 dA + 2\int_{A_{1p}} y dA\right],$$

or:

$$M = R_e b\left(h^2 - \frac{c^2}{3}\right), \quad (6)$$

and equation (5) is

$$M_p = R_e Z = R_e bh^2. \quad (7)$$

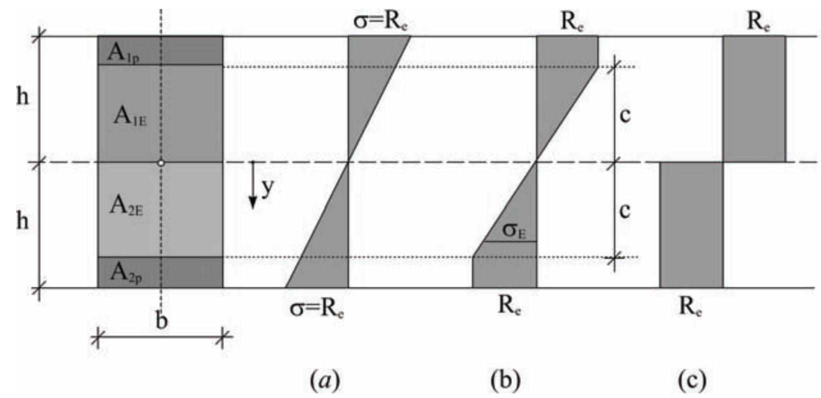

Fig. 2. Stress distributions in rectangular beams
(*a*) elastic, (b) elastic – plastic, (c) maximum plastic

The cross section of I beams possesses two axis of symmetry, just as is the case with the rectangular one. Neutral axis in the case of pure bending caps with the main central axis during all the stress stages. Bending moment in elastic – plastic stage (Fig. 3. (b)) is, then:

$$M = 2\int_{A_{1E}} \sigma_E \, y\, dA + 2\int_{A_{1p}} R_e \, y\, dA, \tag{8}$$

$$M = R_e\left[be(2h - e) + a(h - e)^2 - \frac{1}{3}ac \right]. \tag{9}$$

At the moment of fully plastified stress distribution, yield moment has the value of:

$$M_p = R_e\left[be(2h - e) + a(h - e)^2 \right]. \tag{10}$$

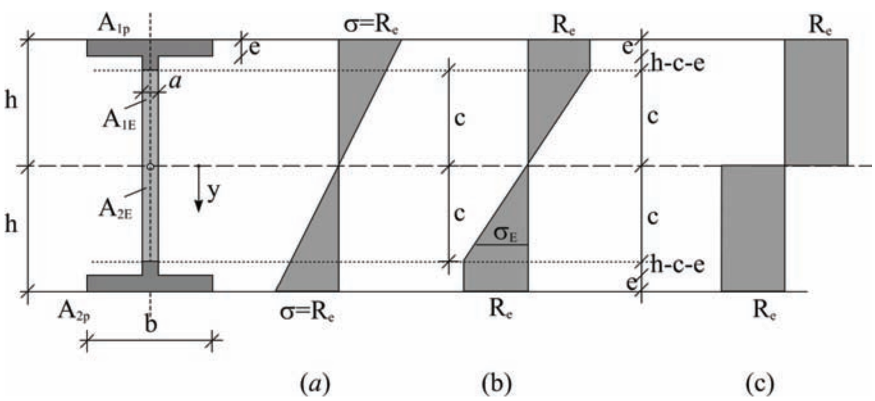

Fig. 3. Stress distribution in I cross section beam
(a) maximum elastic M=Me, (b) elastic – plastic, (c) maximum plastic M=Mp

3. Combined Axial and Bending Forces

Let us focus now on an arbitrary beam cross section of perfectly elastic – plastic material, that is exposed to the complex bending, which means combined effect of bending and tension.

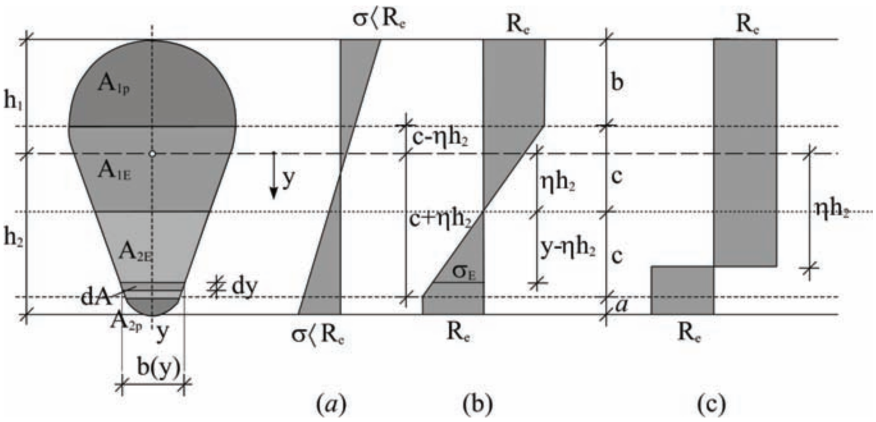

Fig. 4. Combined bending and axial stresses
(*a*) elastic stress distribution, (b) partially plastic stress distribution, (c) fully plastic stress distribution

For sufficiently small strains, the stress will be everywhere elastic and hence will be proportional to the strain as shown in Fig. 4. (*a*). However, as the loads are increased, the stress at the outermost fiber will reach the yield stress R_e. For slightly higher loads, a plastic region will form, as indicated in Fig. 4. (b). Finally, the section will become fully plastic, as indicated in Fig. 4. (c).
As per definition, the forces at the cross section are as follows:

$$M = \int_A \sigma y \, dA \quad \text{and} \quad N = \int_A \sigma \, dA.$$

The stress resultants in elastic – plastic state (Fig. 4. (b)), are given by:

$$M = R_e \left[\frac{1}{c} \left(\int_0^c y^2 b(y) dy + \int_0^c y^2 b(y) dy \right) - \int_{-h_1}^{-(c-\eta h_2)} y b(y) dy + \int_{c+\eta h_2}^{h_2} y b(y) dy \right], \tag{11}$$

$$N = R_e \left[\frac{1}{c} \left(\int_0^c y b(y) dy - \int_0^c y b(y) dy \right) + \left(\int_{-h_1}^{-(c-\eta h_2)} b(y) dy - \int_{c+\eta h_2}^{h_2} b(y) dy \right) \right]. \tag{12}$$

The stress resultants in the fully plastic state, presented at (Fig. 4. (c)), can be expressed as follows:

$$M = R_e \left(-\int_{-h_1}^{\eta h_2} y b(y) dy + \int_{\eta h_2}^{h_2} y b(y) dy \right), \tag{13}$$

$$N = R_e \left(\int_{-h_1}^{\eta h_2} b(y) dy - \int_{\eta h_2}^{h_2} b(y) dy \right). \tag{14}$$

Depending on the value of axial force parameter "η", obtains various values. Thus, we can obtain various combinations of M and N forces, which can lead then to complete plastification of cross section. Fig. 4. gives an obvious representation that the parameter η will have different values, depending on the axial force attribute. If that axial force is actually the pressure force, then:

$$0<\eta h_2 \le h_2,$$

and if the axial force actually stands for tension force, then we get

$$0<\eta h_1 \le h_1.$$

If axial force were to be neglected, the section would have a yield moment M_p (Exp. (5)) whereas in simple tension without bending the yield force is:

$$N_p = R_e A, \tag{15}$$

where A stands for cross section area.
It is convenient to define dimensionless variables by:

$m=M/M_p$ and $n= N/N_p$.

If we exclude the η parameter from (13) and (14) equations and if we introduce m and n in it, it is possible to construct, within the system of orthogonal m and n axes, the curve which would determine which combination of bending and axial force brings about the full plastic state of the cross section.
In order to be specific, we shall first discuss the rectangular section presented at Fig. 5. The stress resultants at elastic – plastic state, are based on (11) and (12), and given by:

$$M = R_e b\left[h^2 - \frac{c^2}{3} - \eta^2 h^2\right], \tag{16}$$

$$N = 2R_e b\eta h. \tag{17}$$

The stress resultants in the fully plastic state, based on (13) and (14), are:

$$M = R_e bh^2\left(1-\eta^2\right), \tag{18}$$

$$N = 2R_e b\eta h. \tag{19}$$

Yield moment for rectangular cross section is expressed in (7), and the yield force is, evidently:

$$N_p = 2R_e bh. \quad (20) \tag{20}$$

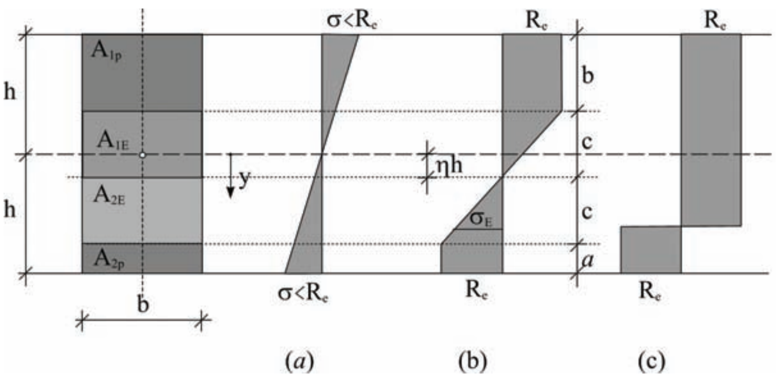

Fig. 5. Combined bending and axial stresses in rectangular beam
(*a*) elastic stress distribution, (b) partially plastic stress distribution,
(c) fully plastic stress distribution

If we exclude η parameter from (18) and (19) expressions, and taking care of (7) and (20), we obtain:

$$\frac{M}{M_p} = 1 - \left(\frac{N}{N_p}\right)^2, \tag{21}$$

or, according to the previously introduced determinants:

$$m = 1 - n^2 . \tag{22}$$

Equation (22) can be given a simple geometrical interpretation in a plane with coordinates m and n. Indeed, the resulting curve is the parabola (Fig. 6.) with the horizontal axis, which fulfills all the conditions n=±1, m=0 and n=0, m=±1. The curve defines the combination of internal forces which cause the full plastification of rectangular section.

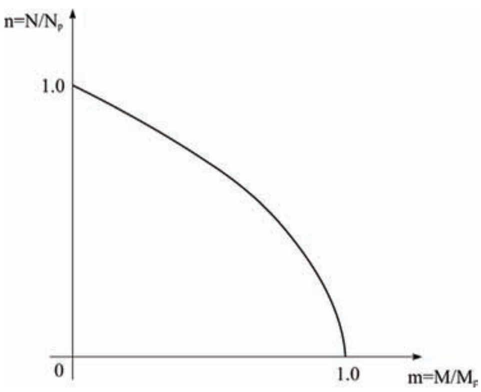

Fig. 6. Interaction curve for combined tension and bending – rectangular beam

Any point of interaction curve corresponds to a fully plastic section of the beam. Also, it is evident that for a point outside the interaction curve no distribution of stresses can be found which will not exceed the yield stress. Finally, points inside the curve represent stress distributions which are less than fully plastic. Therefore, the interaction curve may also be called the yield curve of the section.
In case of cross section of I beam it is necessary to consider two cases in respect to the position of the neutral axis in the cross section.

1. Neutral axis is in the cross section web

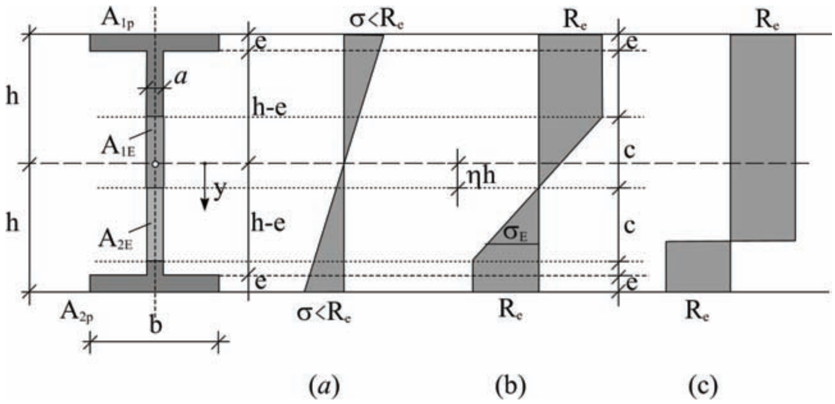

Fig. 7. Combined bending and axial stresses in I beam
(*a*) elastic stress distribution, (b) partially plastic stress distribution,
(c) fully plastic stress distribution

The stress resultants in the elastic plastic state, presented at Fig. 7. (b), are given by:

$$M = R_e\left[b_e\left(2h - e\right) + a\left(h - e\right)^2 - a\eta^2h^2 - \frac{1}{3}ac^2\right], \tag{23}$$

$$N = 2R_e a\eta h. \tag{24}$$

The stress resultants in the fully plastic state, presented at Fig. 7. (c), are given by:

$$M = R_e\left[be\left(2h - e\right) + a\left(h - e\right)^2 - a\eta^2h^2\right], \tag{25}$$

$$N = 2R_e a\eta h. \tag{26}$$

Yield moment for I shape cross section is defined by the expression (10), and yield axial force is then:

$$N_p = R_eA = R_e\left[2be + 2a\left(h - e\right)\right]. \tag{27}$$

We use the same procedure, as it is the case with rectangular beam, where we get the interactive dependency of m and n for I beam and for the position of the neutral axis in the cross section web, which is then:

$$m = 1 - n^2\frac{A^2}{4aZ}, \tag{28}$$

where A stands for the cross section area and Z is plastic modulus

2. Neutral axis is in the cross section flange

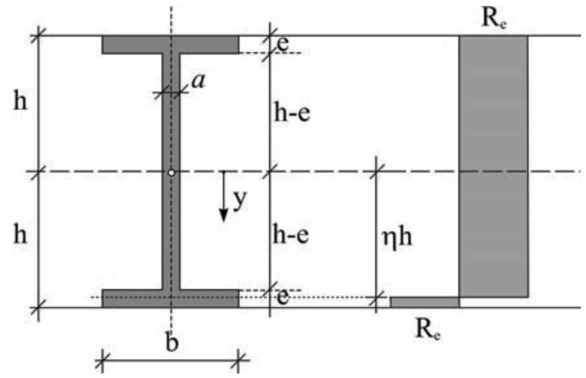

Fig. 8. Combined bending and axial stresses in I beam – fully plastic stress distribution

The stress resultants in the fully plastic state, (Fig. 8.), are given by:

$$M = R_ebh^2\left(1 - \eta^2\right), \tag{29}$$

$$N = R_e\left[2b\left(e - h + \eta h\right) + 2a\left(h - e\right)\right], \tag{30}$$

and, the interactive relation is then:

$$m = \frac{A(1-n)\left[h - A(1-n)\frac{1}{4b}\right]}{Z}. \tag{31}$$

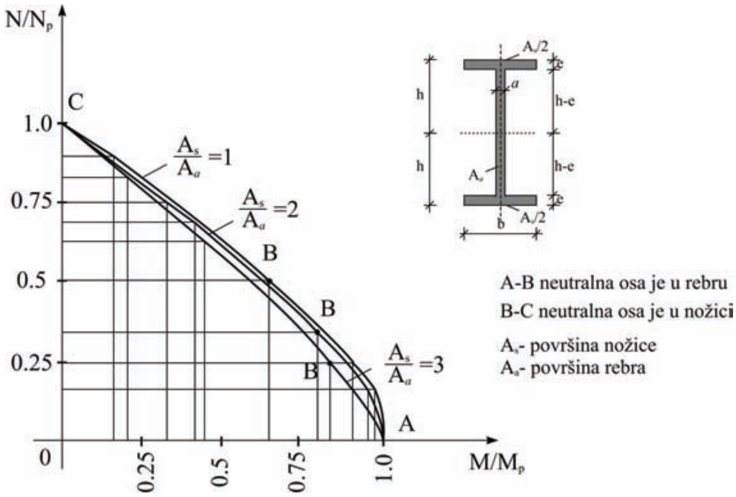

Fig. 9. Interaction curves for combined tension and bending – I beam

Fig. 9. presents interaction curves for I beam and ratio between web area and flange area 1,2 and 3. Sections A – B on the curves presented in the picture correspond to the position of the neutral axis in the cross section web, whereas sections B – C correspond to the position of the neutral axis in the cross section flange.

4. Beams under Pure Shear

In case of I beam under pure shear in elastic state, the distribution of the vertical stresses in the web creates a shallow parabolic shape, whereas in the flanges there are certain tangent horizontal shear stresses, distributed as shown in Fig. 10. [1].

Vertical shear stresses in the flanges can be neglected, so it is possible to allow for the web to take over all shear stresses up to the middle of the flange width. That is how we would get the appropriate approximate value of maximum shear stresses in the longitudinal section of the web, by dividing the shear force with the effective web area:

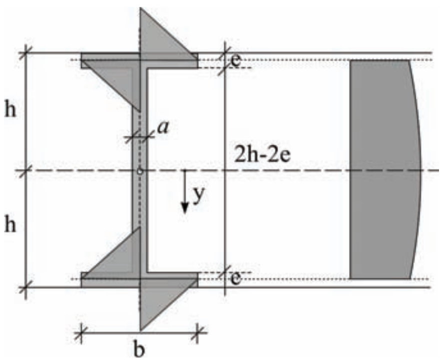

Fig. 10. I beam under pure shear

$$\tau_{max} = \frac{T}{A_a} = \frac{T}{a(2h - 2e)}. \tag{32}$$

Numerous experiments show that the influence of transverse forces can be completely neglected at rectangular beams, as well as at I beams. Rather small plastic deformations caused by the shear and present in the web have no significant influence on the value of the yield moment which remains unchanged until complete plastification of the web, meaning until the value of shear forces fulfills the following condition:

$$T < \frac{R_e A_a}{\sqrt{3}}. \tag{33}$$

Shear force, which cause the fully plastic state of the web of I cross section in the case of pure shear, yield shear force, is:

$$T_p = \frac{R_e A_a}{\sqrt{3}}, \tag{34}$$

and it is determined using yield condition due to Mises, which, for beams under pure shear is:

$$3\tau^2 \le R_e^2.$$

For rectangular cross section of b x 2h dimensions, and if Zuravski formula and parabolic distribution of shear stresses along the height of the cross section in elastic, elastic – plastic and fully plastic state have been accepted, then the value of the shear force can be expressed as follows:

$$T = \int_A \tau \, dA = \frac{2}{3} \tau b 2h = \frac{4}{3} \tau b h. \tag{35}$$

According to (35), shear stress will, then, have the value

$$\tau = \frac{3}{4} \frac{T}{bh}.$$

Yield shear force of rectangular cross section can be defined by the expression (35) and Mises yield condition as:

$$T_p = \frac{4 R_e bh}{3\sqrt{3}}. \tag{36}$$

5. BEAMS UNDER COMBINED STRESS

Former analysis dills with determining of limit bearing capacity of cross sections in case of clear bending, eccentric tension and pure shear. In this chapter the yield curves in terms of moment and axial force, as well as in terms of moment, axial and shear force in the case of rectangular beam and I beam, which are loaded in only one plane, are determined. In analysis Mises yield condition has been used, which for beams under pure bending, as well as combined bending and axial forces is:

$$\sigma^2 \le R_e^2, \tag{37}$$

while in the case of combined shear and bending of beams, Mises yield condition states:

$$\sigma^2 + 3\tau^2 \le R_e^2. \tag{38}$$

5.1. Rectangular cross section

– the influence of bending and axial forces

Using the existing expressions for bending moment, axial force, yield moment and yield axial force of rectangular cross section, (b x 2h):

$$M = \sigma b h^2 \left(1-\eta^2\right), \qquad N = 2\sigma b \eta h,$$
$$M_p = R_e b h^2, \qquad N_p = 2R_e b h,$$

stress function has the form of:

$$\sigma^2 - \sigma \frac{M}{bh^2} - \frac{N}{4b^2h^2} = 0. \tag{39}$$

The real root of this equation, using Mises yield condition (37), is:

$$\sigma = \frac{M}{2bh^2} + \left(\frac{M^2}{4b^2h^4} + \frac{N^2}{4b^2h^2}\right)^{1/2} = R_e. \tag{40}$$

Yield curve in terms of $m = \dfrac{M}{M_p}$ and $n = \dfrac{N}{N_p}$ is obtained as:

$$\frac{m}{2} + \left(\frac{m^2}{4} + n^2\right)^{1/2} = 1. \tag{41}$$

This curve is being analyzed in Capter 3. and represented in Fig. 6.

– the influence of bending, normal and shear forces

Using the yield condition (38), the expression for stress function (40) and the expression for shear stress of rectangular beam (b x 2h) under pure shear, we would get the function of combined influences in the form of:

$$\left[\frac{M}{2bh^2} + \left(\frac{M^2}{4b^2h^4} + \frac{N^2}{4b^2h^2}\right)^{1/2}\right]^2 + 3\left(\frac{3}{4}\frac{T}{bh}\right)^2 = R_e^2. \tag{42}$$

Introducing dimensionless values $m = \dfrac{M}{M_p}$, $n = \dfrac{N}{N_p}$ and $t = \dfrac{T}{T_p}$, where M_p, N_p and T_p are already determined (eq. (18), (19), (36)), the yield curve can be obtained as:

$$\frac{m^2}{2}+n^2+m\left(\frac{m^2}{4}+n^2\right)^{1/2}+t^2=1. \tag{43}$$

5.2 I cross section

– the influence of bending and axial forces

Taking into consideration already derived expressions for resultant forces (25) and (26) for I beam (dimensions as in Fig. 3.) the stress function is obtained as:

$$\sigma^2-\frac{M}{be(2h-e)+a(h-e)^2}\sigma-\frac{N^2}{4a\left[be(2h-e)+a(h-e)^2\right]}=0. \tag{44}$$

The real root of this equation, using yield condition (37) is:

$$\sigma=\frac{1}{2}\frac{M}{be(2h-e)+a(h-e)^2}+\left(\frac{M^2}{4\left[be(2h-e)+a(h-e)^2\right]^2}+\frac{N^2}{4a\left[be(2h-e)+a(h-e)^2\right]}\right)^{1/2}=R_e. \tag{45}$$

With regard to expressions (10) and (27) for yield moment and yield axial force for I beam, the function of combined influences of bending and axial forces on limit state of cross section is obtained as the form of:

$$\frac{m}{2}+\left(\frac{m^2}{4}+n^2\frac{\left[2be+2a(h-e)\right]^2}{4a\left[be(2h-e)+a(h-e)^2\right]}\right)^{1/2}=1, \tag{46}$$

that represents the interaction curve presented in Fig. 9.

– the influence of the bending, axial and shear forces

The stress function for the case of combined stresses can be obtained, using yield condition (38) and stress function (45), in the form of:

$$\left[\frac{1}{2}\frac{M}{be(2h-e)+a(h-e)^2}+\left(\frac{M^2}{4\left[be(2h-e)+a(h-e)^2\right]^2}+\frac{N^2}{4a\left[be(2h-e)+a(h-e)^2\right]}\right)^{1/2}\right]^2$$

$$+3\left(\frac{T}{a(2h-e)}\right)^2=R_e^2. \tag{47}$$

Substituting already derived expressions for yield moment (10), yield axial force (27) and the yield shear force (34), into equation (47), the function of combined influences of bending, axial and shear forces is defined by:

$$\frac{m}{2}+m\left(\frac{m^2}{4}+n^2\frac{A^2}{4AZ}\right)^{1/2}+n^2\frac{A^2}{4AZ}+t^2=1, \tag{48}$$

where A and Z represent geometric characteristics of the cross section that have been previously defined, and m, n and t relations have been previously stated, as well.

6. CONCLUSION

The problem of the determination of limit bearing capacity of beam cross section is considered in this paper. The limit load capacity of beams under pure bending, eccentric tension, pure shear, as well as combined stress was analyzed.

Using the presumptions about the elastic – plastic material, its homogeneity, isotropic and incompressibility features and Mises yield conditions, we have derived the influence functions of the bending moment and axial force, as well as the bending moment, axial and shear force on the described cross section limit bearing capacity.

REFERENCES

1. Massonet Ch. et Save M.: Calcul plastique des constructions, Vol 1, Structures dependent d`un parametre, Centro belge-luxemburgeois d`information de l'acier, Bruxelles, 1967.
2. Massonet Ch. et Save M.: Calcul plastique des constructions, Vol 1, Structures dependent d`un parametre, Centro belge-luxemburgeois d`information de l`acier, Bruxelles, 1967.
3. Mijalković M.: Granična nosivost linijskih sistema, magistarski rad, strana 134, Građevinski fakultet u Nišu, Niš, 1992.
4. Philip G., Hodge Jr Ph.D: Plastic analysis of structures, McGrow-Hill company, New York, Toronto, London, 1959.
5. Лазаров Љ.: Гранична носивост на линијски конструкции, Скопје, 2000.
6. Horne M.R.: Plastic Theory of Structures, The M.I.T. Press, Cambridge, 1971.
7. Moy S.S.J.: Plastic Methods of Steel and Concrete Structures, Macmillan Press Ltd, 1996.
8. Sreten Stevanović, Marina Mijalković: Granična nosivost poprečnog preseka grednog nosača, Monografija "Modeliranje proračun i ispitivanje konstrukcija", strana 266-276, Institut za Građevinarstvo i arhitekturu Građevinskog fakulteta u Nišu, Niš, 1995.
9. Marina Mijalković: Granična nosivost linijskih nosača, Zbornik radova Građevinskog fakulteta u Nišu N° 13/14, strana 63-70, Niš, 1992/93.
10. Marina Mijalković, Ljubinka Nikodijević-Božilov: Granična analiza kontinualnih nosača, Zbornik radova Građevinskog fakulteta u Nišu N° 15/16, strana 125-132, Niš, 1994/95.
11. Ljubinka Nikodijević-Božilov, Marina Mijalković: One Procedure for Determination of Limit Loading of Rectangular Highed Frames of One Bay with Movable Joints, Časopis "Facta Universitatis", Series "Architectures and Civil Engineering", Vol. 1, N° 2, strana 187-195, University of Niš, 1995.
12. Ljubinka Nikodijević-Božilov, Marina Mijalković: One Procedure for Determination of Limit Loading of Built-in Portal Frames of One Bay with Movable Joints, Časopis "Facta Universitatis", Series "Architectures and Civil Engineering", Vol. 1, N° 3, strana 381-392, University of Niš, 1996.

GRANIČNA ANALIZA KOMBINOVANO NAPREGNUTIH GREDA

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Ovaj rad razmatra problem utvrđivanja granične nosivosti preseka grede pod čistim savijanjem, ekscentričnim zatezanjem, čistim smicanjem, kao i kombinovanim naprezanjem. Izvedene su uticajne funkcije momenta savijanja i aksijalne sile, kao i momenta savijanja, aksijalne i sile smicanja na graničnu nosivost preseka u slučaju greda pravougaonih i I profila.