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DETERMINATION OF THE CHARACTERISTIC PARAMETERS IN THE GENERAL COLLINEAR SPACES IN THE GENERAL CASE

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Abstract. *In order to perform mapping in the general-collinear spaces which are uniformly (biunivocally) determined with five pairs of associated points, it is necessary to determine the following parameters in the general case: the vanishing plane, space axis (principal normal lines), foci, if there are any (apices of the identical rays of straight lines) and directrix planes, if there are any (appended identical fields of points).Construction methods which can be used in this paper are the Descriptive and projective geometry methods. The paper is primarily based on the constructive treatment of general collinear spaces in the general case, in whose course a pair of Monge's projections are used.*

Key words: *General-collinear spaces, vanishing planes, axes, foci and directrix spatial.*

1. INTRODUCTION

"Two general-collinear spaces are considered collinearly associated if in every point of a space a point of another space is associated, to every plane of a space a plane of another space is associated, and to every point in a plane of the former space, a point of the latter space belongs to the associated plane in this space. This definition implies that each straight line in a plane in the former space will have a straight line in the latter space associated, and each point on the straight line and every straight line in a plane in the former space will have a point associated on the associated straight line in the associated plane in the latter space.

The consequence of the previous definition and its effect is, that in the associated pair of planes of two collinear spaces, two collinearly associated fields would occur, and the further consequence of this is that all the associated series of points and all the associated pencils of straight lines in two collinear spaces ought to be projectively associated.

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2. Division of General Collinear Spaces

Two general-collinear spaces can be given in the most general manner by two groups of 5 points (planes) $\mathbf{A_1}$, $\mathbf{B_1}$, $\mathbf{C_1}$, $\mathbf{D_1}$, $\mathbf{E_1}$ (α_1, β_1, χ_1, δ_1, ε_1) in the space θ^1 and $\mathbf{A_2}$, $\mathbf{B_2}$, $\mathbf{C_2}$, $\mathbf{D_2}$, $\mathbf{E_2}$ (α_2, β_2, χ_2, δ_2, ε_2) in the space θ^2, so that four points do not lie in the plane, out of them three do not lie on one straight line, (four planes do not pass by one point, three of them do not pass by one straight line). In the paper, the general-collinear (OK) spaces are given with five pairs of biunivocally associated points. In this way the series of points lying on the associated straight lines are associated, then the pencils of planes through the associated straight lines as tufts, piles of straight lines through the associated points, piles of planes through the associated points and fields of points, which are determined by the foursome of associated points in the associated planes.

With the aid of these projective elements in the OK spaces, the characteristic parameters can be constructively determined.

1. vanishing planes
2. space axes (principal normal lines)
3. space foci (apexes of associated identical piles of straight lines)
4. directrix planes (associated identical fields of points)

On the basis of the conclusion "In the general case, the general collinear fields do not have identical fields, and therefore there will be no foci in these spaces" – doctoral dissertation A. Jovanovic, one may divide the issue into two basic cases of the OK spaces:

1. General case - OK spaces without foci or directrix planes
2. Special case - OK spaces with foci and directrix planes

The first case is when the OK are given in the most general way, with the aid of a set of five biunivocally associated points.

The second case is when the OK spaces are given so that the foursome apex in the first space, obtained out of three basic points, and the point of penetration of the junction of the remaining two points through the plane of the first three, is identical or similar to the associated foursome apex obtained in the same way in the second space (doctoral dissertation – A. Jovanovic).

3. General Case of the OK Spaces – Without Foci and Directrix Planes

Two OK spaces θ^1 and θ^2 are given in a pair of Monge's projections with the five pairs of biunivocally associated points, $\mathbf{A_1}$, $\mathbf{B_1}$, $\mathbf{C_1}$, $\mathbf{D_1}$ and $\mathbf{E_1}$ in the space θ^1 and $\mathbf{A_2}$, $\mathbf{B_2}$, $\mathbf{C_2}$, $\mathbf{D_2}$ and $\mathbf{E_2}$ in the space θ^2. In this way the projective relationship of two spaces has been established.

3.1. Determination of the vanishing planes in the OK spaces in the general case

Each projective space has one infinitely distant plane. The fictitious plane of the first space is associated to the vanishing plane of the second space and vice versa. The constructive procedure for the determination of the vanishing planes comprises establishing the double proportion on the associated series of points.

The vanishing plane $\mathbf{N_1}$ (Fig. 1) in the space θ^1 associated to the infinitely distant plane $\mathbf{N_2}^\infty$ of the space θ^2, is determined by the vanishing points $\mathbf{T_1}$, $\mathbf{H_1}$ and $\mathbf{L_1}$ on the series of points ($\mathbf{A_1B_1}$), ($\mathbf{C_1D_1}$) and ($\mathbf{C_1B_1}$). The vanishing plane $\mathbf{M_2}$ (fig.1) in the space θ^2 associated to the infinitely distant plane $\mathbf{M_1}^{\infty}$ of the space θ^1, is determined by the vanishing points $\mathbf{R_2}$, $\mathbf{P_2}$ and $\mathbf{Q_2}$ on the series of points ($\mathbf{A_2B_2}$), ($\mathbf{C_2D_2}$) i ($\mathbf{C_2B_2}$).

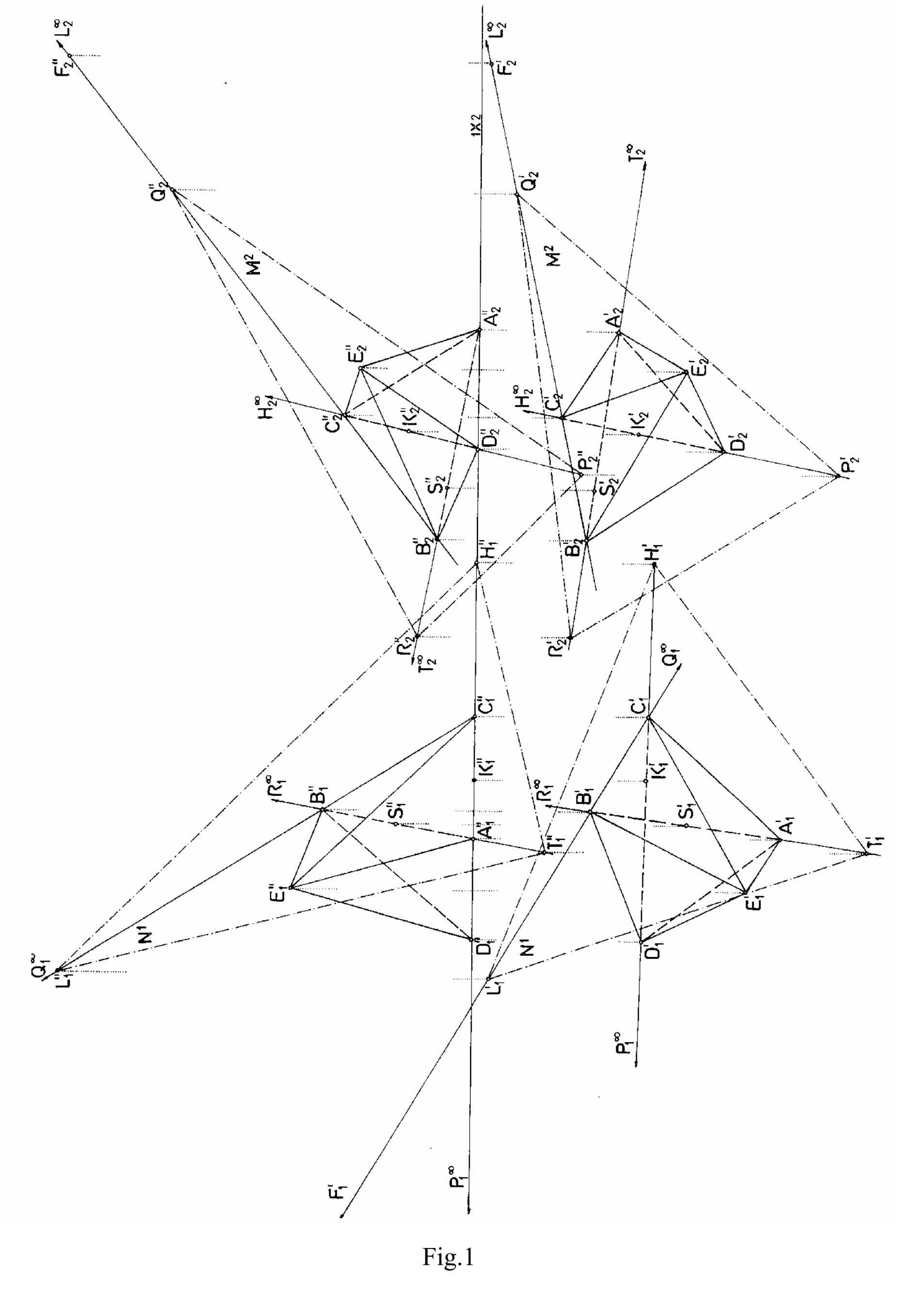

Fig.1

In order to determine the vanishing points on the associated series of pints, it is necessary to previously find yet another associated point on each carrier of the series, because it is then possible to establish the double proportions. The constructive procedure for the determination of the vanishing points is to bring the carriers of the series in the perspective position by coinciding of one associated pair of points.

The straight line $\mathbf{A_1B_1}$ penetrates the plane $\mathbf{C_1D_1E_1}$ in the point $\mathbf{S_1}$ of the space θ^1, to which the point $\mathbf{S_2}$ is associated, which is a penetration point of the straight line $\mathbf{A_2B_2}$ through the plane $\mathbf{C_2D_2E_2}$ in the space θ^2. To the infinitely distant point $\mathbf{R_1}^\infty$ on the series ($\mathbf{A_1B_1}$) the point $\mathbf{R_2}$ on the series ($\mathbf{A_2B_2}$) is associated, so the vanishing point $\mathbf{R_2}$ is determined from the relationship λ=($\mathbf{A_1B_1S_1R_1}^\infty$)=($\mathbf{A_2B_2S_2R_2}$). In the same way the vanishing points $\mathbf{P_2}$ and $\mathbf{Q_2}$ on the series ($\mathbf{C_2D_2}$) and ($\mathbf{C_2B_2}$) are determined. The point $\mathbf{K_1}$ is the penetration of the straight line $\mathbf{C_1D_1}$ through the plane $\mathbf{A_1B_1E_1}$ in the space θ^1, and the point $\mathbf{K_2}$ associated to it is the penetration of the straight line $\mathbf{C_2D_2}$ through the plane $\mathbf{A_2B_2E_2}$ in the space θ^2. The vanishing point $\mathbf{P_2}$ on the series ($\mathbf{C_2D_2}$) is determined with the aid of double proportion λ=($\mathbf{C_1D_1K_1P_1}^\infty$)=($\mathbf{C_2D_2K_2P_2}$). The penetration of the straight line $\mathbf{C_1B_1}$ through the plane $\mathbf{A_1D_1E_1}$ is the point $\mathbf{F_1}$ in the space θ^1, to which the point $\mathbf{F_2}$ is associated, the penetration $\mathbf{C_2B_2}$ through the planed $\mathbf{A_2D_2E_2}$ in the space θ^2. To the point $\mathbf{Q_2}$ on the series ($\mathbf{C_2B_2}$), a fictitious point $\mathbf{Q_1}^\infty$ on the series ($\mathbf{C_1B_1}$) is associated, and it was determined from the relationship λ=($\mathbf{C_1B_1F_1Q_1}^\infty$)=($\mathbf{C_2B_2F_2Q_2}$).

The vanishing plane $\mathbf{N_1}$ ($\mathbf{T_1H_1L_1}$) in the space θ^1, which is associated to the infinitely distant plane of the space θ^2, is determined with the vanishing points: $\mathbf{T_1}$ on the series ($\mathbf{A_1B_1}$), $\mathbf{H_1}$ on the series ($\mathbf{C_1D_1}$) and $\mathbf{L_1}$ on the series ($\mathbf{C_1B_1}$). The vanishing point $\mathbf{T_1}$, which is associated to the fictitious point $\mathbf{T_2}^\infty$ on the series ($\mathbf{A_2B_2}$), is determined with the aid of the double proportions λ=($\mathbf{A_1B_1S_1T_1}$)=($\mathbf{A_2B_2S_2T_2}^\infty$). The point $\mathbf{H_1}$, which is associated to the fictitious point $\mathbf{H_2}^\infty$ on the series ($\mathbf{C_2D_2}$), determined from the relationship λ=($\mathbf{C_1D_1K_1H_1}$)=($\mathbf{C_2D_2K_2H_2}^\infty$). The vanishing point $\mathbf{L_1}$, which is associated to the fictitious point $\mathbf{L_2}^\infty$ on the series ($\mathbf{C_1B_1}$), is determined with the aid of double proportion λ=($\mathbf{C_1B_1F_1L_1}$)=($\mathbf{C_2B_2F_2L_2}^\infty$).

The vanishing planes, (Fig. 1), $\mathbf{N_1}$ in the space θ^1 and $\mathbf{M_2}$ in the space θ^2, are presented in the pair of Monge's projections given by the triangles $\mathbf{T_1H_1L_1}$ i $\mathbf{R_2P_2Q_2}$, and in further work, in order to have a simpler construction for the determination of the penetration of the straight line through the plane, that is the radiating position of the vanishing planes, the trails of plans are used, the first $\mathbf{n_1}$, $\mathbf{m_1}$ and the second $\mathbf{n_2}$, $\mathbf{m_2}$ (Fig. 2).

3.2. Determination of the axes (principal normal lines) in the OK spaces in the general case

The infinitely distant point $\mathbf{W_2}$ in the vanishing plane $\mathbf{M_2}$ of the space θ^2 is associated to the infinitely distant point $\mathbf{W_1}^\infty$ of the normal lines $\mathbf{n_1}$ on the vanishing plane $\mathbf{N_1}$ in the space θ^1. The $\mathbf{W_2}$ point is called the center of the space θ^2 (stand of the principal normal) and that is a point through which a straight line $\mathbf{o_2}$ passes and it is perpendicular to the vanishing plane $\mathbf{M_2}$ and is called the axis (principal normal line) of the space θ^2.

To the fictitious point $\mathbf{X_2}^\infty$ no the normal lines $\mathbf{m_2}$ on the vanishing plane $\mathbf{M_2}$ in the space θ^2 an infinitely distant point $\mathbf{X_1}$ is associated in the vanishing plane $\mathbf{N_1}$ of the space θ^1. The point $\mathbf{X_1}$ is called the center of the space θ^1 (the stand of the principal normal line) and it is the point through which a straight line, $\mathbf{o_1}$, passes, which is perpendicular to the vanishing plane $\mathbf{N_1}$ and is called an axis (principal normal line) of the space θ^1.

These two straight lines, $\mathbf{o_1}$ and $\mathbf{o_2}$, are the only straight lines which are, prior and after the mapping, perpendicular to the corresponding vanishing planes, and projectively associated, and are called the axes of the space.

3.2.1. Constructive procedure for determination of the centers in the OK spaces in the general case

Through the straight line $\mathbf{A_1B_1}$, of the space θ^1, (Fig. 2), a plane $\beta_1 \perp \mathbf{N_1}$ is set, β_1($\mathbf{n_{1A}}$ through the point $\mathbf{A_1}$ and $\mathbf{n_{1B}}$ through the point $\mathbf{B_1}$), which is penetrated by the straight line $\mathbf{C_1D_1}$ in the point $\mathbf{J_1}$. From the relationship $\lambda=(\mathbf{C_1D_1J_1P_1}^\infty)=(\mathbf{C_2D_2J_2P_2})$, the point $\mathbf{J_2}$ on the series ($\mathbf{C_2D_2}$) is determined. The straight line $\mathbf{n_{1A}}$ set with the point $\mathbf{A_1}$ intersects the cutting line $\mathbf{q_1}= \mathbf{B_1J_1}$, of the β_1 plane and $\mathbf{B_1C_1D_1}$, in the point $\mathbf{V_1}$. The fictitious point of the straight line $\mathbf{q_1}$ is marked with $\mathbf{O_1}^\infty$. With the aid of double proportion $\lambda=(\mathbf{B_1J_1V_1O_1}^\infty)=(\mathbf{B_2J_2V_2O_2})$ the point $\mathbf{V_2}$ is determined, on the straight line $\mathbf{q_2}= \mathbf{B_2J_2}$, where the point $\mathbf{O_2}$, is the penetration $\mathbf{q_2}$ through the vanishing plane $\mathbf{M_2}$ of the space θ^2.

The junction $\mathbf{n_{2A}}=\mathbf{A_2V_2}$, which is associated to the straight line $\mathbf{n_{1A}}=\mathbf{A_1V_1}$, penetrates the vanishing plane $\mathbf{M_2}$ in the point $\mathbf{W_2}$, center of the space θ^2. Through the center $\mathbf{W_2}$ a straight line, $\mathbf{o_2}$, passes and it is perpendicular to the vanishing plane $\mathbf{M_2}$ and is called the axis (principal normal) of the space θ^2.

In the same way, the straight line $\mathbf{o_1}$ is determined, which is an axis of the space θ^1, and is associated to the straight line $\mathbf{o_2}$ axis of the space θ^2.

Through the straight line $\mathbf{C_2D_2}$, of the space θ^2, (Fig. 2), a plane $\alpha_2 \perp \mathbf{M_2}$ is set, α_2 ($\mathbf{m_{2C}}$ through the point $\mathbf{C_2}$ and $\mathbf{m_{2D}}$ through the point $\mathbf{D_2}$). The plane α_2, is penetrated by the straight line $\mathbf{A_2B_2}$ in the point $\mathbf{Z_2}$. Out of the relationship $\lambda = (\mathbf{A_2B_2Z_2T_2}^\infty) = (\mathbf{A_1B_1Z_1T_1})$, the point $\mathbf{Z_1}$ on the series ($\mathbf{A_1B_1}$) is determined in the space θ^1. The straight line $\mathbf{m_{2D}}$ set through the point $\mathbf{D_2}$ intersects the cutting line $\mathbf{s_2}$, of the plane α_2 and $\mathbf{A_2B_2C_2}$ in the point $\mathbf{U_2}$. On the straight line $\mathbf{s_2}$ the fictitious point is $\mathbf{I_2}^\infty$. With the aid of the relationship $\lambda = (\mathbf{C_2Z_2U_2I_2}^\infty) = (\mathbf{C_1Z_1U_1I_1})$, the point $\mathbf{U_1}$ is determined on the straight line $\mathbf{s_1} = \mathbf{C_1Z_1}$, where the point $\mathbf{I_1}$ is the penetration point $\mathbf{s_1}$ through the vanishing plane $\mathbf{N_1}$, of the space θ^1.

The junction $\mathbf{m_{1D}} = \mathbf{D_1U_1}$, which is associated to the straight line $\mathbf{m_{2D}} = \mathbf{D_2U_2}$ penetrates the vanishing plane $\mathbf{N_1}$ in the point $\mathbf{X_1}$, which is a center of the space θ^1. Through the center $\mathbf{X_1}$ the straight line, $\mathbf{o_1}$, passes, which is perpendicular to the vanishing plane $\mathbf{N_1}$ and is called the axis (principal normal line) of the space θ^1.

The axes of the space are the only two straight lines which are perpendicular to the corresponding vanishing planes, projectively associated, and the points on them can be mapped only with the aid of double proportion (Fig. 2).

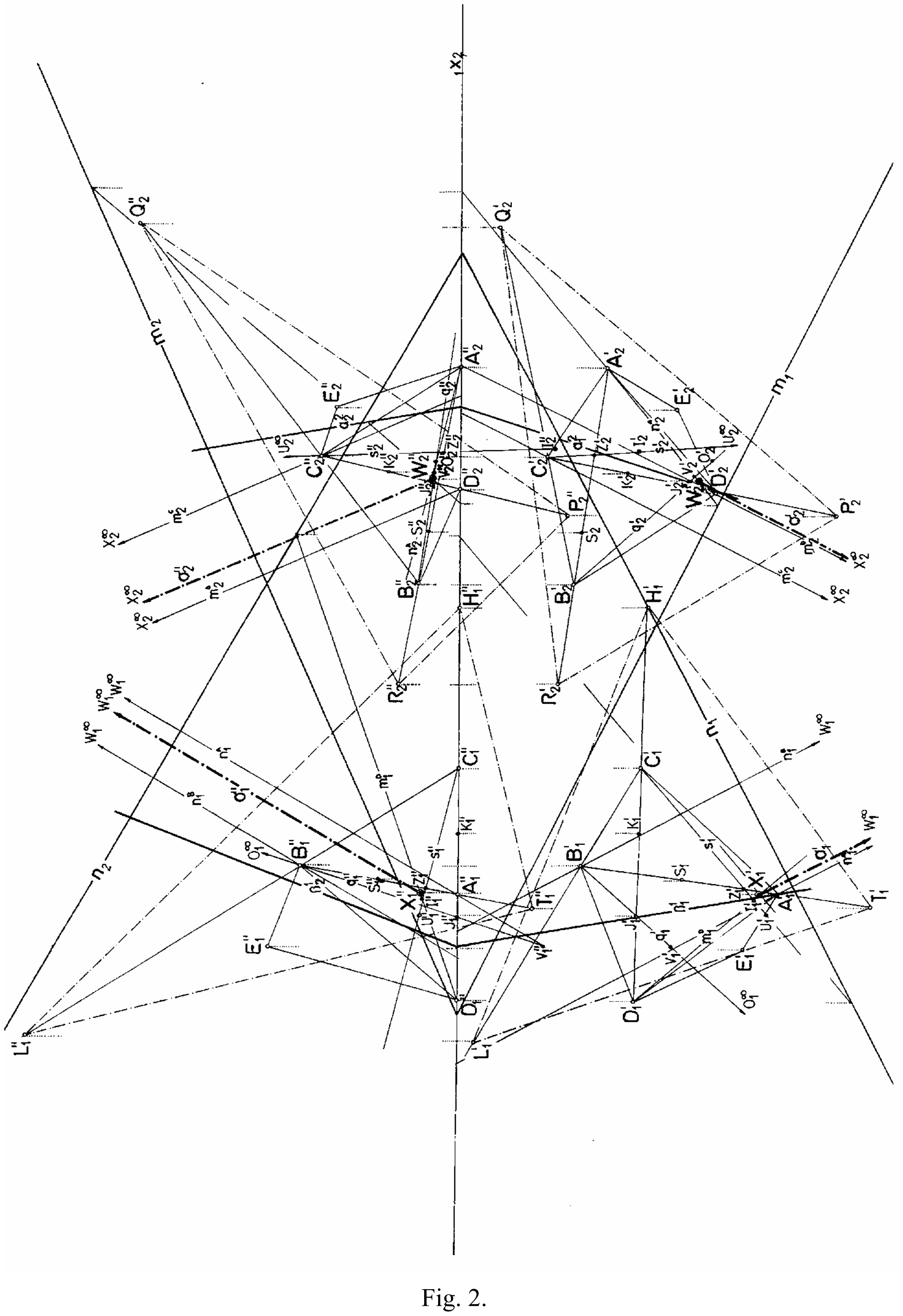

Fig. 2.

3.3. Determination of the directrix planes in the OK spaces in the general case

If though any of the points and axis $\mathbf{o_1}$ of the space θ^1, a rotating circle ψ_1 is placed, with the radius $\mathbf{r_1}$, it will be associated, in the space θ^2, with a rotating cone ψ_2, whose apex $\mathbf{W_2}$, is the axis $\mathbf{o_2}$, and it passes through the associate point and vice versa. All the planes of the space θ^1, parallel to the vanishing plane $\mathbf{N_1}$ intersect the circular cylinder ψ_1 across the circumferences $\Sigma\,\mathbf{t_1}$, of the radius $\mathbf{r_1}$. These planes are parallel to the vanishing plane $\mathbf{N_1}$ in the space θ^1, are associated to the planes parallel to the vanishing plane $\mathbf{M_2}$ in the space θ^2, which intersect the cone across the circumferences $\Sigma\,\mathbf{t_2}$. Only two planes from the set of circumferences $\Sigma\,\mathbf{t_2}$ will intersect the cone ψ_2, across the circumferences of the radius $\mathbf{r_1}$ and vice versa. We call these planes the directrix planes and the fields of points are identical in them.

3.3.1. Constructive procedure for determination of the directrix planes in the OK spaces

Through the points $\mathbf{A_1}$, $\mathbf{B_1}$ and $\mathbf{E_1}$ of the space θ^1, (Fig. 3), the planes α_1,β_1 are ε_1 set, parallel to the vanishing plane $\mathbf{N_1}$. The axis of the space $\mathbf{o_1}$, penetrates these planes in the points $\overline{A_1}$, $\overline{B_1}$ and $\overline{E_1}$. To them the planes are associated, through the points $\mathbf{A_2}$, $\mathbf{B_2}$ and $\mathbf{E_2}$ in the space θ^2, α_2, β_2 and ε_2, parallel to the vanishing plane $\mathbf{M_2}$. The axis of the space $\mathbf{o_2}$, penetrates these planes in the points $\overline{A_2}$, $\overline{B_2}$ and $\overline{E_2}$. The points $\overline{A_1}$, $\overline{B_1}$, $\overline{E_1}$ and $\overline{A_2}$, $\overline{B_2}$, $\overline{E_2}$ are mutually associated.

For this constructive procedure, the vanishing planes are brought into the radiating position, via transformation, thus the planes which are parallel in them, where the cutting circumferences of the circular cylinders and cones are situated, are seen in radiating position.

Through the point $\mathbf{A_1}$ and the axis $\mathbf{o_1}$, the rotating circle ψ_1^A is set, which the planes parallel to the vanishing $\mathbf{N_1}$, intersect on the circumferences $\Sigma\mathbf{t_1}^A$, whose radius is $\mathbf{r_1}^A$, and its true size is determined by the rotation. To the cylinder ψ_1^A the rotation cone ψ_2^A in the space θ^2 is associated, whose apex is $\mathbf{W_2}$, the axis $\mathbf{o_2}$ and it passes through the associated point $\mathbf{A_2}$. The radius of the rotating cone ψ_2^A circumference in the plane α_2, is $\mathbf{r_2}^A$. The planes parallel to the vanishing plane $\mathbf{M_2}$, of the space θ^2, intersect the cone ψ_2^A on the circumferences $\Sigma\mathbf{t_2}^A$. Two planes, from the set $\Sigma\mathbf{t_2}^A$, which intersect the cone ψ_2^A on the circumferences of the radius $\mathbf{r_1}^A$ are the directrix planes $\mathbf{K_2}^A$ and $\mathbf{Q_2}^A$, of the space θ^2. To them, the directrix planes $\mathbf{K_1}^A$ and $\mathbf{Q_1}^A$ are associated in the space θ^1.

The penetrations of the axes $\mathbf{o_1}$ and $\mathbf{o_2}$, of the spaces $\theta^1\,\theta^2$, through the directrix planes $\mathbf{K_1}^A$, $\mathbf{Q_1}^A$ and $\mathbf{K_2}^A$, $\mathbf{Q_2}^A$, are the points $\mathbf{Y_1}^A$ and $\mathbf{Z_1}^A$ in the space θ^1 and $\mathbf{Y_2}^A$ and $\mathbf{Z_2}^A$ in the space θ^2. Out of the relationship $\lambda=(\mathbf{W_2X_2}^{\infty}\mathbf{Y_2Z_2})=(\mathbf{W_1}^{\infty}\mathbf{X_1Y_1Z_1})$ the points $\mathbf{Y_1}^A$ and $\mathbf{Z_1}^A$ in the space θ^1 on the axis $\mathbf{o_1}$ can be determined, and thus the positions of the directrix planes $\mathbf{K_1}^A$ and $\mathbf{Q_1}^A$.

If in the space θ^2, through the point $\mathbf{A_2}$ and the axis $\mathbf{o_2}$, a rotating circle ξ_2^A is set, then it has associated the rotating cone ξ_1^A, through the apex $\mathbf{X_1}$, axis $\mathbf{o_1}$ and point $\mathbf{A_1}$, of the space θ^1. Just as in the previously described procedure, the same positions of the directrix planes $\mathbf{K_1}^A$ and $\mathbf{Q_1}^A$ in the space θ^1 are obtained as well as $\mathbf{K_2}^A$ and $\mathbf{Q_2}^A$ in the space θ^2, which is a proof of the biunivocally performed collinear mapping.

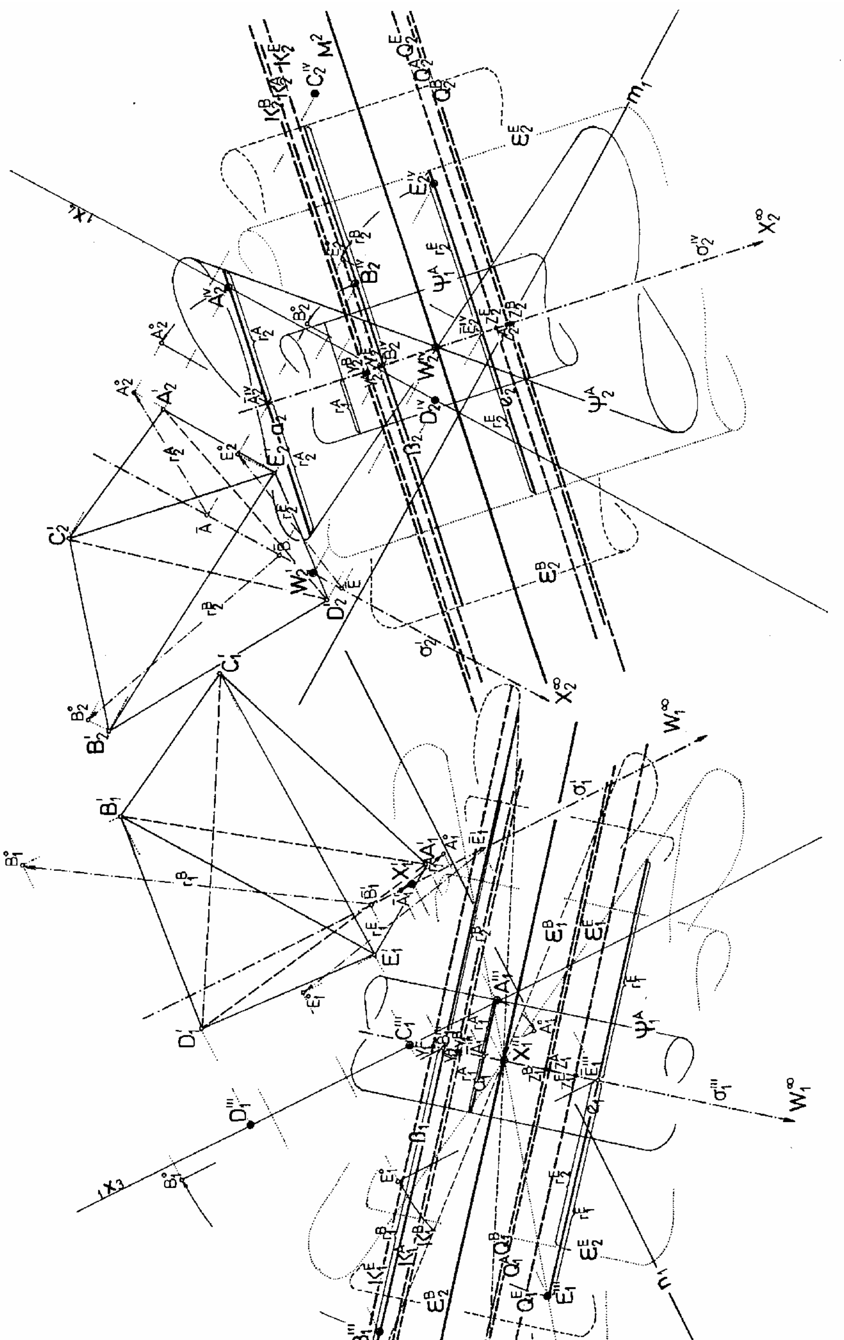

Fig.3.

If the same procedure for determination of the directrix planes is conducted for the points $\mathbf{B_1}$ and $\mathbf{E_1}$ in the space θ^1 and for the points $\mathbf{B_2}$ and $\mathbf{E_2}$ in the space θ^2, the positions of the directrix planes, $\mathbf{K_1^B}$, $\mathbf{Q_1^B}$ and $\mathbf{K_2^B}$, $\mathbf{Q_2^B}$; $\mathbf{K_1^E}$, $\mathbf{Q_1^E}$ and $\mathbf{K_2^E}$, $\mathbf{Q_2^E}$, are not the same as for the points $\mathbf{A_1}$ and $\mathbf{A_2}$. Even the axis penetration points, $\mathbf{o_1}$ and $\mathbf{o_2}$, through these directrix planes do not coincide $\mathbf{Y_1^A} \neq \mathbf{Y_1^B} \neq \mathbf{Y_1^E}$ i $\mathbf{Z_1^A} \neq \mathbf{Z_1^B} \neq \mathbf{Z_1^E}$ as well as $\mathbf{Y_2^A} \neq \mathbf{Y_2^B} \neq \mathbf{Y_2^E}$ and $\mathbf{Z_2^A} \neq \mathbf{Z_2^B} \neq \mathbf{Z_2^E}$.

On the basis of the previously said, the conclusion is made that the OK spaces given in the most general form do not have the directrix planes (identical fields of points), and thus no foci of the space.

4. Conclusion

In order to perform mapping in the general-collinear spaces which are uniformly (bi-univocally) determined with five pairs of associated points, it is necessary to determine the characteristic parameters: the vanishing planes and axes of space. The vanishing planes are associated to the infinitely distant planes of the space, so with their aid the fictitious elements can be brought into the finiteness. The axes of the space are the only two associated straight lines which are perpendicular to the corresponding vanishing planes.

In the general form of the OK spaces there are neither foci (the apexes of the associated identical piles of straight lines), nor the directrix planes (associated identical fields of points), which is proven as well in this paper, with whose aid the mapping in the OK spaces in the general case is rendered simpler. For the determination of the projective creations in the general case of the OK spaces, the vanishing planes and axes of the space will be used, the new parameters will be the subject of further investigation, with whose aid the association of two collinear spaces will be possible.

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ODREDJIVANJE KARAKTERISTIČNIH PARAMETARA U OPŠTE-KOLINEARNIM PROSTORIMA U OPŠTEM SLUČAJU

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Da bi se moglo vršiti preslikavanje u opšte-kolinearnim prostorima koji su zadati sa pet parova jednoznačno (biunivoko) pridruženih tačaka, potrebno je da se u opštem položaju odrede sledeći karakteristični parametri: nedogledne ravni, ose prostora (glavne normale), žiže, ako ih ima (temena pridruženih identičnih svežnjeva pravih) i direktrisne ravni, ako ih ima (pridružena identična polja tačaka). Konstruktivne metode koje će se koristiti u ovom radu su metode Nacrtne i Projektivne geometrije. Rad se pretežno bazira na konstruktivnoj obradi opšte-kolinearnih prostora u najopštijem slučaju, pri čemu se koristi par Monžovih projekcija.