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PLANE SECTION OF CONE AND CYLINDER IN COMPUTER GEOMETRY

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Abstract. *In this paper a mathematical apparatus for determination of plane section of cone and cylinder was formed. By using the descriptive geometric approach, the contour lines of these quadrics were determined. The fact that the tangent lines of a circle could be transformed to the tangent lines of an ellipse using affinity was employed. In that way surfaces are represented by contour lines (tangent lines of basic ellipse in oblique projection) and thus they have a realistic view. Intersecting plane α is a plane normal to a frontal plane. For determination of intersecting points of intersecting curve between the plane α and the quadrics, the lock of auxiliary planes, which contain the vertex of quadrics, was used. Each auxiliary plane from the observed lock intersect the surface in two lines which intersect the given plane α in two points. By using a sufficient number of auxiliary planes the intersecting curve as a set of pairs of points for all auxiliary planes is determined and the intersecting curve was drawn by lightening of these pairs of points on the graphical screen.*

Key words: *quadrics, plane section, computer geometry.*

1. Contour Lines in Oblique Projection by Using Polarity and Affine Transformation

1.1. Affinity

If a basic circle of a surface is in horizontal plane then its oblique projection is an ellipse which is obtained by the affine transformation of the circle. The horizontal plane H of the first orthogonal projection and the horizontal plane H_K of the oblique projection are in affine correlation where axis of affinity is coincident to x axis and this is the intersecting line between these two planes. In figure 1 the connection between the first orthogonal and the first oblique projection of point A (x_A, y_A, z_A) is presented.

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Dimensions of an axis y_k can be equal to dimensions of an axis *y*, longer or shorter in comparison to the *y* axis. The most frequent decrease is for a quarter ($y_k : y = 3 : 4$), a third ($y_k : y = 2 : 3$) and a half ($y_k : y = 1 : 2$) .

If angle γ of oblique projection and decrease $\frac{y_k}{y} = \frac{Y_{S_k}}{Y_S}$ are given then it is necessary to obtain the first oblique projection A_k' from the point A. The position of point A_k' could be found (Fig.1) in the coordinate system *Oxy* by using the following equations (Obradovic, 1997, Cveticanin, 1998)

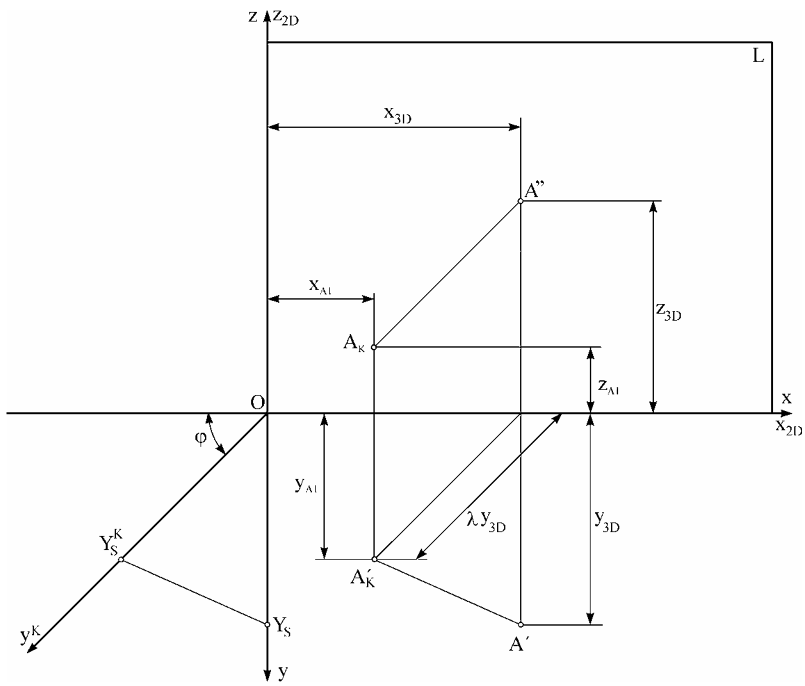

Fig. 1. Point coordinates in a pair of orthogonal projections and in an oblique projection

$$y_{A_k} = \frac{y_A \cdot Y_{Sk}}{Y_S} \tag{1}$$

$$x_{A_1} = x_A - y_{A_k} \cdot \cos\gamma \tag{2}$$

$$y_{A_1} = y_{A_k} \cdot \sin\gamma \tag{3}$$

Equations (1),..., (3) affinely transform all points from the first orthographic to the first oblique projection.

1.1.1. Points in oblique projection and in a pair of orthographic projections

Point A in the oblique projection is defined by three coordinates (x_A, y_A, z_A) and in plane *Oxz* it can be defined by two coordinates (x_{A_1}, z_{A_1}). We can obtain a transformation from 3D to 2D coordinate system by using the equation (2) and the following equation

$$z_{A_1} = z_A - y_{A_1} \tag{4}$$

1.1.2. Affine transformation of basic circle

Basic circle is given with centre C (x_C, y_C, 0) and radius R (Fig. 2). Parametric equations of circle in plane *Oxy* are

$$x_b = x_c + R \cdot \cos \varphi$$

$$y_b = y_c + R \cdot \sin \varphi$$

where parameter φ is between 0^0 and 360^0 (360 circle points are used).

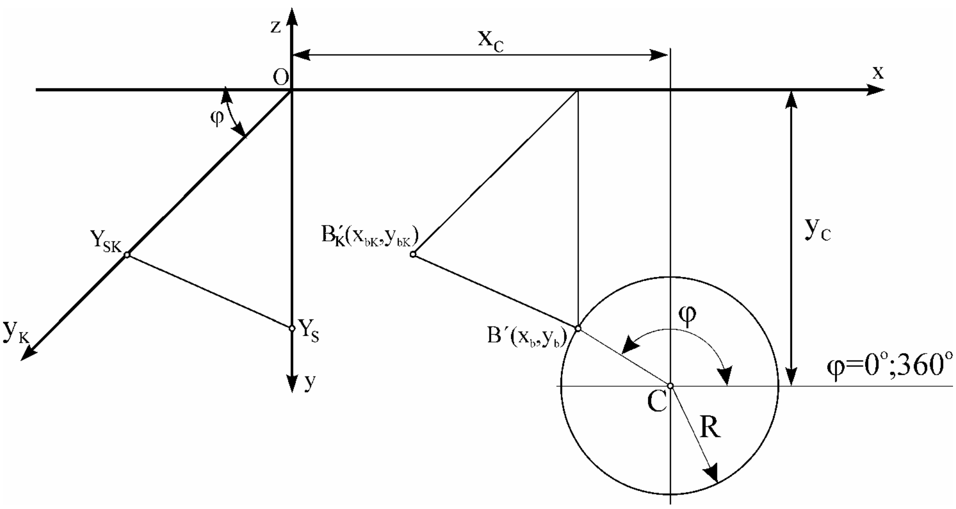

Fig. 2. Affine transformation of circle

1.2. Polarity

Determination of contour lines of cone and cylinder is based on constructing of tangent lines from the given point P to the circle K (Fig.3).

These tangent lines (t_1, t_2) are touching the circle K in points K_1, K_2 and the polar line q (with pole P) is a connecting line between these two points. When pole P is at infinity then the polar line is coincident with the circle diameter.

Four points A, Q, B and P on the diametric line PC are in harmonic relation

$$(PQAB) = -1$$

or
$$\frac{[PB]}{[QB]}:\frac{[PA]}{[QA]}=-1 \tag{5}$$

Starting from equation (5) it is possible to determine the coordinates of points K_1, K_2. Namely, for the given local coordinate system Cx_1y_1, the points P, A, and B have the coordinates (Obradovic, 1997)

$$P = P(x_{1_p},0,0);\ A = A(-R,0,0);\ B = B(R,0,0)$$

where

$x_{1_p} = [PC]$ – Length of segment PC; R – Radius of circle K

Centre C of circle K in the same coordinate system is given with $C = C(0,0,0)$. If the length of segment QC is designated with f then other lines are defined with relations

$$[QB] = R - f \tag{6}$$

$$[PA] = -([BP] + 2\cdot R) = -([CP] + R) = -(R + \sqrt{x_{1_p}^2 + y_{1_p}^2}) \tag{7}$$

$$[QA] = -(R + f) \tag{8}$$

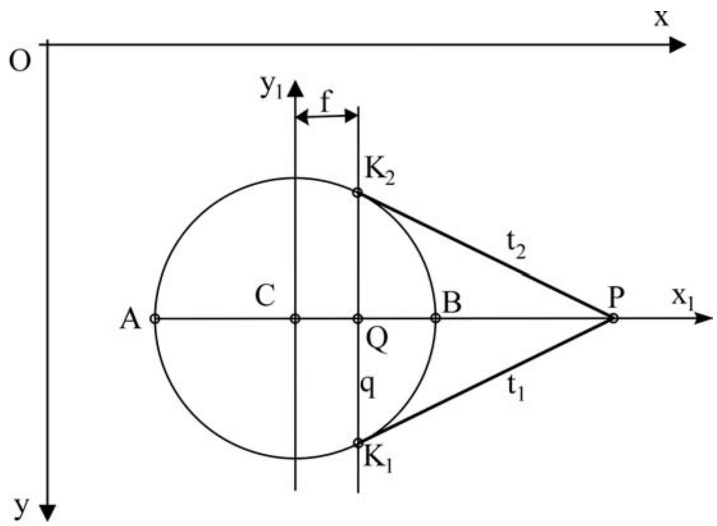

Fig. 3. Polarity

By including equations (6), (7) and (8) into equation (5) one obtains

$$\frac{[PB]}{R-f}:\frac{[PB]+2\cdot R}{R+f}=-1 \tag{9}$$

Based on Fig.3 one can find

$$[PB] = [CP] - R = \sqrt{x_{1_p}^2 + y_{1_p}^2} - R = -[PB] \tag{10}$$

By substitution of (10) for (9) the following equation is obtained

$$\frac{\sqrt{x_{1_p}^2 + y_{1_p}^2} - R}{R - f} = \frac{\sqrt{x_{1_p}^2 + y_{1_p}^2} + R}{R + f}$$

From this equation one can find value f

$$f = \frac{R^2}{\sqrt{x_{1_p}^2 + y_{1_p}^2}} \tag{11}$$

Coordinates of points P and C in global coordinate system Oxy are given with

$$P = P(x_p, y_p, 0),\ C = C(x_c, y_c, 0)$$

Then in the same system

$$[CP] = x_{1p} = \sqrt{(x_p - x_c)^2 + (y_p - y_c)^2}$$

Equation (11) in the global coordinate system is

$$f = \frac{R^2}{\sqrt{(x_p - x_c)^2 + (y_p - y_c)^2}} \tag{12}$$

When the circle is in the frontal plane then the value f is given with equation

$$f = \frac{R^2}{\sqrt{(x_p - x_c)^2 + (z_p - z_c)^2}} \tag{13}$$

where the points P and C in the system Oxz are given with the coordinates

$$P = P(xp, 0, zp);\ C = C(xc, 0, zc)$$

2. CONTOUR LINES OF CONE IN OBLIQUE PROJECTION

Centre of basic circle is $C\,(x_c, y_c, 0)$, radius is R and vertex of cone is point $V\,(x_v, y_v, z_v)$ (Fig. 4). Point C is transformed by affine transformation into C_K, circle is transformed into ellipse and point V into V_K. Contour lines of cone in oblique projection are tangent lines of ellipse from point V_K. This pair of tangent lines of ellipse is transformed into tangent lines of circle from point V_H (in figure 4 point V_H is obtained by steps 1, 2, 3). The contact points K_1, K_2 are affinely transformed into points K_1, K_2 and these tangent lines of ellipse are coincident in oblique projection with the contour lines of the cone.

Analytical interpretation of this construction is given with equation (1),...,(4) which are giving us two coordinates of vertex of cone in plane Oxz, i.e. $V_{Hk}(x_{v_1}, z_{v_1})$. In the next step one can find the coordinates of points V_H in the system Oxz. Line p_1 is parallel to axis y_k, i.e. angle between x axis and this line is $tg\,\gamma$. Line p_1 contains point V_{Hk} and according to that, this line intersects z axis in point

$$b_{p_1} = z_{v_1} - tg\gamma \cdot x_{v_1}$$

This line intersects x axis in point $M(x_{V_H}, 0)$ where

$$x_{v_H} = -\frac{b_{p_1}}{tg\gamma}$$

Line p_2 is parallel to line $Y_{S_K}Y_S$ and a referent line for scaling can be line CC_k too (Fig. 4). In system Oxz points C, C_k are defined with $C(x_C, -y_C), C_k(x_{C_1}, -y_{C_1})$. Equation of line p_2 is

$$z = a_{p_2}x + b_{p_2} \tag{14}$$

By substitution of coordinates of points C, C_k in equation (14) and by solving of this pair of given equations we can find direction coefficient of this line

$$a_{p_2} = \frac{y_{C_1} - y_C}{x_C - x_{C_1}}$$

By substitution of coordinates of point $V_{H_k}(x_{V_1}, z_{V_1})$ into (14) we obtain

$$b_{p_2} = z_{V_1} - a_{p_2}x$$

Point V_H is determined with coordinates x_{V_1}, z_{V_1} where is

$$z_{V_H} = a_{p_2}x_{V_H} + b_{p_2}$$

Equation (12) or (13) will give us the value f

$$f = \frac{R^2}{\sqrt{(x_C - x_{V_H})^2 + (y_C + z_{V_H})^2}}$$

Let us designate with p the connecting line between points V_H and C, whose equation in Oxz coordinate system is given with

$$z = a_{p_0}x + b_{p_0} \tag{15}$$

By substitution of coordinates of points $V_H(x_{V_H}, z_{V_H})$ and $C(x_C, -y_C)$ into equation (15) after several steps the following is obtained

$$a_{p_0} = \frac{z_{V_H} + y_C}{x_{V_H} - x_C}$$

where (Fig. 4)

$$\alpha_{p_0} = arctg\, a_{p_0}$$

Polar line q is described with

$$z = a_q x + b_q \tag{16}$$

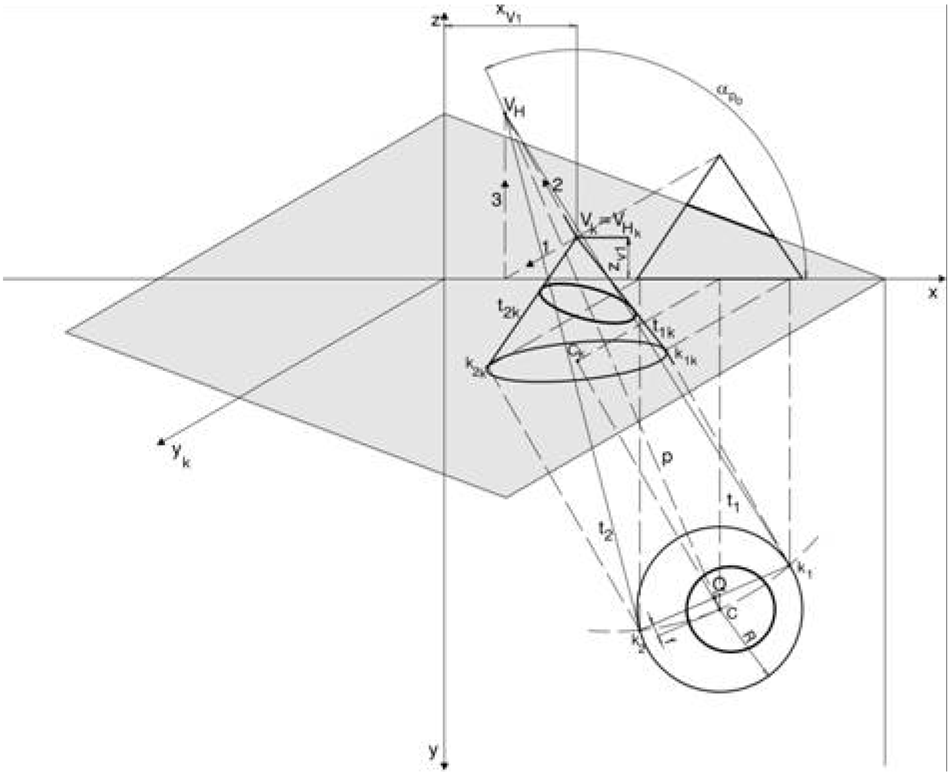

Fig. 4. Contour lines of a cone in oblique projection

and as it is known that this line is orthogonal to the line p then

$$a_q = tg\alpha_q = tg\left(\alpha_{p_0} + \frac{\pi}{2}\right) = -\frac{1}{a_{p_0}}$$

Point Q is the intersecting point between polar line q and line p and in Oxy system it has coordinates

$$x_Q = (L_0 - f)\cos\alpha_{p_{0_1}} + x_{V_H} \qquad y_Q = (L_0 - f)\sin\alpha_{p_{0_1}} - z_{V_H}$$

where

$$\alpha_{p_{0_1}} = arctg\frac{z_{V_H} + y_C}{x_{V_H} + x_C}$$

L_0 is the distance from point V_H to C

$$L_0 = \sqrt{(x_C - x_{V_H})^2 + (y_C + z_{V_H})^2}$$

Value b_q of polar line q on z axis is

$$b_q = -(y_Q + a_q x_Q)$$

Now the intersecting points between polar line q and basic circle K, i.e. points K_1, K_2, can be found.

3. Determination of Intersecting Points of Polar line and Basic Circle

In plane Oxz the equation of circle is

$$(x - x_C)^2 + (z - z_C)^2 = R^2$$

Because $z_c = -y_c$ the last equation is changed into the following equation

$$(x - x_C)^2 + (z + y_C)^2 = R^2 \tag{17}$$

Polar line q is determined by equation (16) and by substituting the equation (16) with (17) one obtains

$$(x - x_C)^2 + (a_q x + b_q + y_C)^2 = R^2$$

Solution of this equation is

$$x_{K_1} = \frac{C_{0_1} + C_{0_2}}{C_{0_3}}\ ;\ x_{K_2} = \frac{C_{0_1} - C_{0_2}}{C_{0_3}}$$

where

$$C_{0_1} = 2(x_C - a_q(b_q + y_C))\ ; \qquad C_{0_2} = \sqrt{C_{0_1}^2 - 4(1 + a_q)^2 C_{0_4}}$$

$$C_{0_3} = 2(1 + a_q^2)\ ; \qquad C_{0_4} = x_C^2 + (b_q + y_C)^2 - R^2$$

and y coordinates of points K_1, K_2 are

$$y_{K_1} = y_C - \sqrt{R^2 - (x_{K_1} - x_C)^2}\ ;\ y_{K_2} = y_C + \sqrt{R^2 - (x_{K_2} - x_C)^2}$$

4. Results

Results given in this paragraph are obtained by using programming system AutoCAD R14 and C++ developing system for AutoCAD Object ARX (AutoCAD Run- time extensions).

Advantages of this method are: figures are presented by using of vector graphics in a known determined ratio, where subsequent usage of figures was possible by some of vector oriented graphical progammes (i.e. CorelDRAW). In all figures the basic circles are transformed into oblique projection by using of 360 points of circle where intersecting curve is defined with 1000 points.

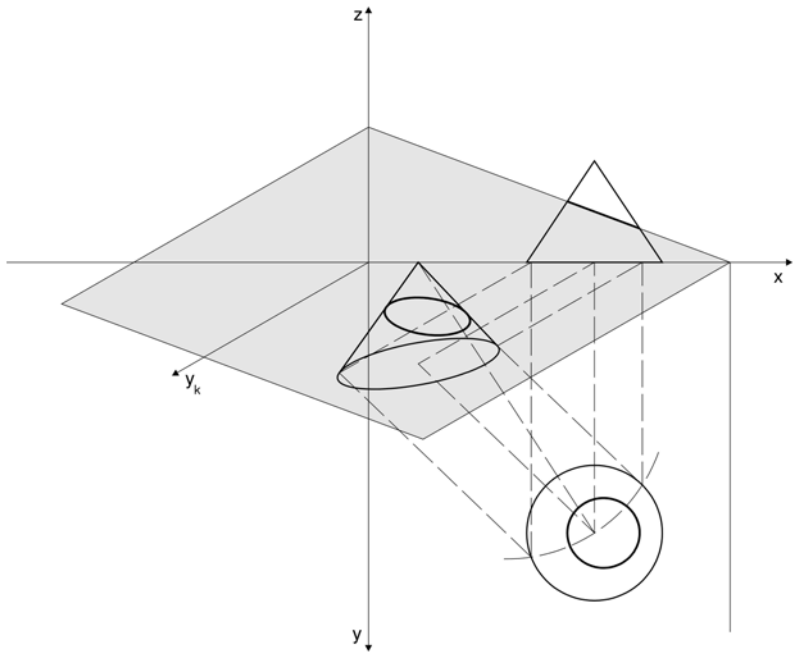

Fig. 5. Plane section of cone - ellipse, $y_k : y = 3 : 4$, $\gamma = 30^o$

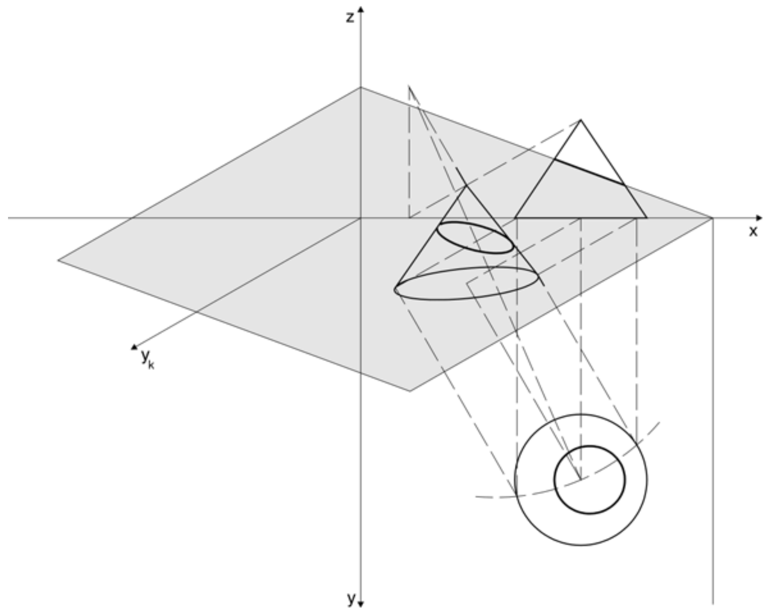

Fig. 6. Plane section of cone - ellipse, $y_k : y = 1 : 2$, $\gamma = 30^o$

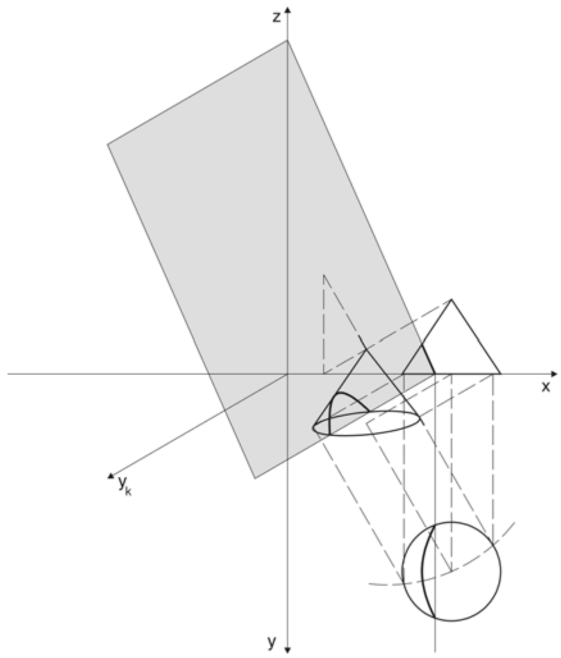

Fig. 7. Plane section of cone - hyperbola, $y_k : y = 1 : 2$, $\gamma = 30^o$

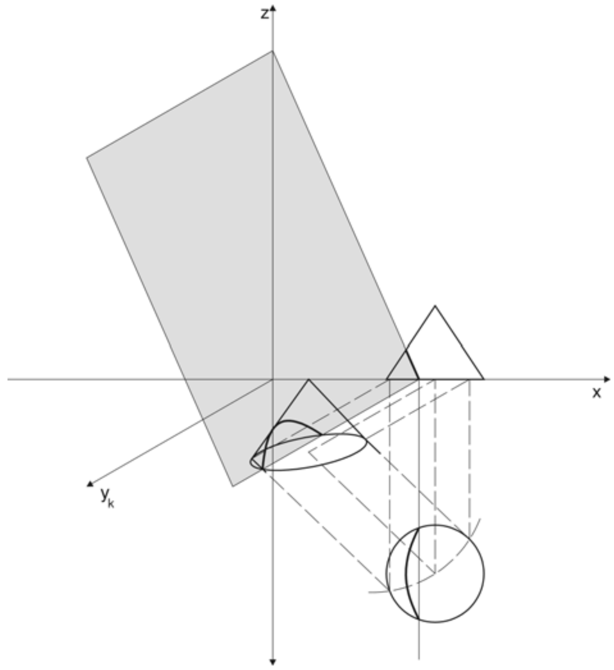

Fig. 8. Plane section of cone - hyperbola, $y_k : y = 3 : 4$, $\gamma = 30^o$

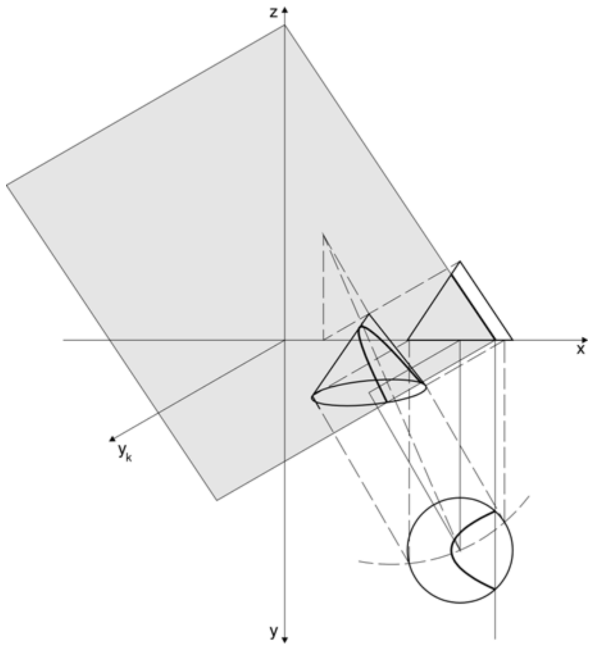

Fig. 9. Plane section of cone - parabola, $y_k : y = 1 : 2$, $\gamma = 30^o$

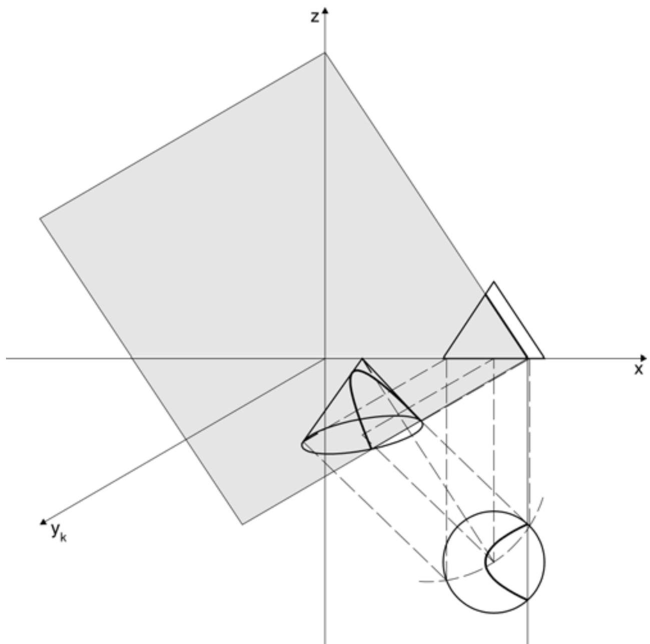

Fig. 10. Plane section of cone - parabola, $y_k : y = 3 : 4$, $\gamma = 30^o$

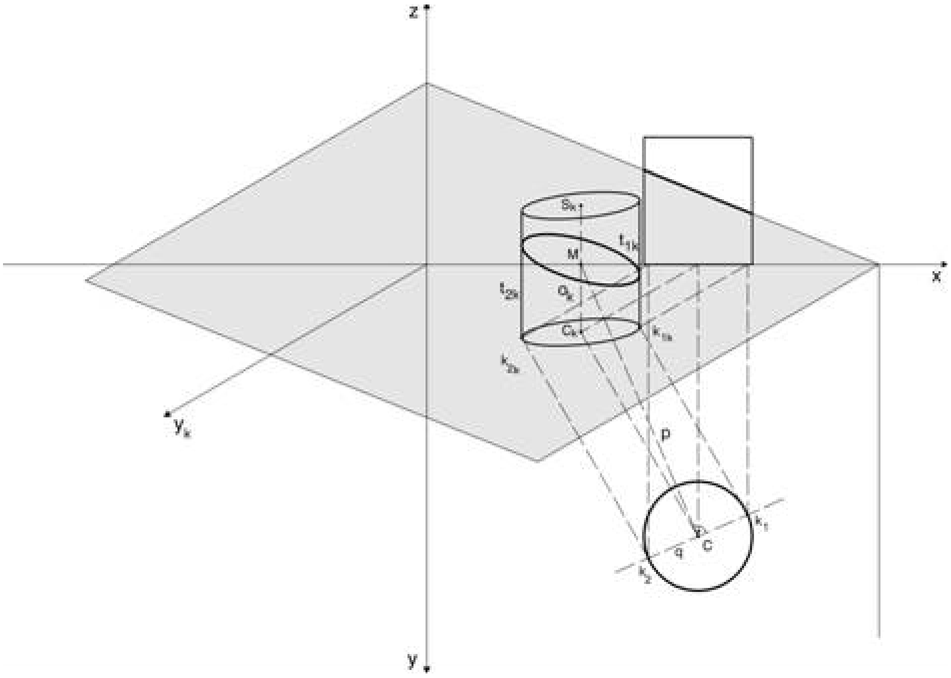

Fig. 11. Plane section of cylinder, $y_k : y = 1 : 2$, $\gamma = 30^o$

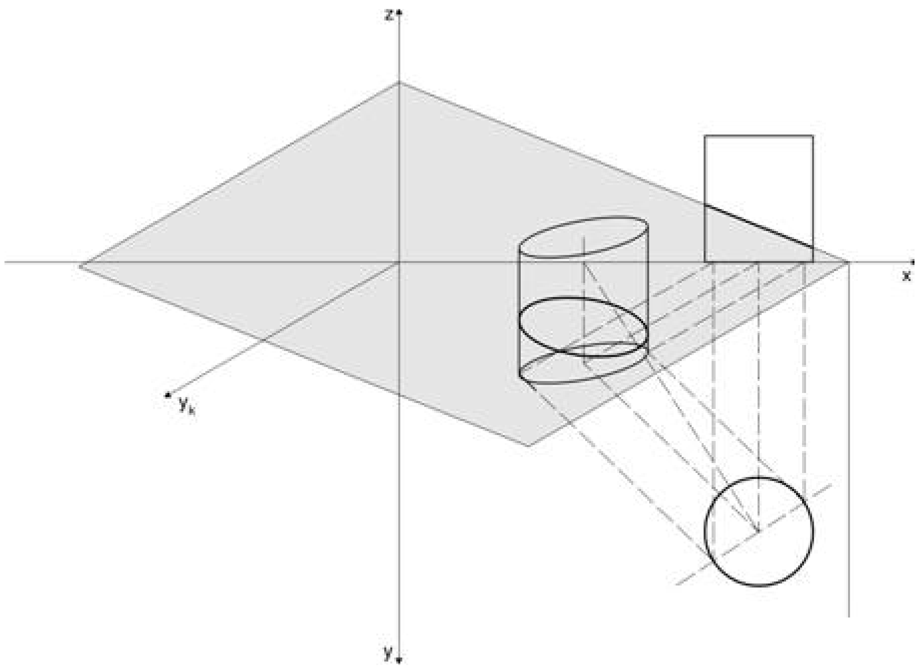

Fig. 12. Plane section of cylinder, $y_k : y = 3 : 4$, $\gamma = 30^o$

4.1. Plane section of the cone

Parameters of the cone are: C(100; 120; 0), H = 45, R = 30. In figures 5 and 6 the intersecting plane is $\alpha(160;\infty; 60)$ and the intersecting curve is an ellipse. In figures 7 and 8 the intersecting curve is the hyperbola and the intersecting plane is $\alpha(202{,}5;\infty;90)$. In figures 9 and 10 the intersecting curve is the parabola and the plane is $\alpha(120;\infty;180)$.

4.2. Plane section of cylinder

The parameters of the cylinder are: C (150;150;0), H = 70, R = 30.

In figures 11 and 12 the plane section of the cylinder is presented and the intersecting plane is $\alpha(250;\infty;100)$.

5. CONCLUSION

The surfaces are presented by contour lines and because of that the surfaces have a realistic view. The intersecting curve is described by a group of 3D points which are determined correctly because the surface is not approximated by using patches.

In the further work it is possible to form similar procedures for determining the plane section of surfaces whose basis is not a circle.

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RAVAN PRESEK KONUSA I CILINDRA U KOMPJUTERSKOJ GEOMETRIJI

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U ovom radu je formiran matematički aparat za određivanje ravnog preseka konusa i cilindra. Korišćenjem nacrtno geometrijskog pristupa određene su konturne izvodnice ovih kvadrika pri čemu je iskorišćena činjenica da se tangente kruga afino preslikavaju u tangente elipse. Na ovaj način površi su prikazane preko konturnih izvodnica (tangenti bazisne elipse u kosoj projekciji) čime su dobile realističan izgled. Ravan koja seče površ je zračna ravan α koja je upravna na frontalnicu. Za određivanja tačaka presečne krive date ravni i kvadrike korišćen je pramen pomoćnih ravni koji sadrži vrh kvadrike. Svaka pomoćna ravan iz posmatranog pramena seše površ po dvema izvodnicama koje opet prodiru zadatu ravan α u dve tačke. Postavljanjem dovoljnog broja pomoćnih ravni presečna kriva se dobija kao skup parova tačaka za sve pomoćne ravni, a sama kriva se crta osvetljavanjem ovog skupa tačaka na grafičkom ekranu.