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GEOMETRY OF STRAIGHT LINES PENCILS

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Milan Janić, Dejan Tanikić

Technical Faculty of Bor
Vojske Jugoslavije 12, 19210 Bor, Serbia and Montenegro

Abstract. *This paper considers a pencil of straight lines in the Euclidean plane as well as the same pencil of straight lines in the projective plane where the projective geometry model M^n is defined with its points forming the sets of* $(n-1)$ *collinear points, whose supporting straight lines belong to the considered pencil of straight lines.*

Key words: $(n-1)$ *collinear points, pencil of straight lines, F planes, AF planes*

1. Introduction

Starting from the projection of a point of polydimensional space, which is represented with the set of n collinear points and the analogous representation of straight line, some properties are derived, and after that a Euclidean plane and euclidean space, which is expanded into the projective plane and projective space. The properties for the elements of one pencil of straight lines are proved, the properties being known as the properties of projective planes.

2. Base concepts

Let $P(P^{12},P^{3},\ldots,P^{n})$ be a point of n-dimensional euclidean space E^n, represented in euclidean plane E^2 with the set of (n-1) collinear points $P^{12},P^{3},\ldots,P^{n}$. Any other point $Q(Q^{12},Q^{3},\ldots,Q^{n})$ of that space is represented in plane with the set of (n-1) collinear points Q^{12}, Q^{3}, …, Q^{n} whose support is a straight line parallel to the support of point P and to the support of points P^{12}, P^{3}, …, P^{n}. We will designate support of point P - straight line p, and support of point Q - straight line q. We will assume that straight lines p and q belong to the pencil of straight lines (O_∞), and that the center of pencil of straight lines is point O_∞.

Definition 2.1. The set of all possible points $X(X^{12},X^{3},\ldots,X^{n})$, where $X^{12}\in P^{12}Q^{12}$, $X^{3}\in P^{3}Q^{3}$, …, $X^{n}\in P^{n}Q^{n}$ represents straight line PQ of pencil of straight lines (O_∞). We

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assume that three points P, Q, R are collinear when three points P^i, Q^i, R^i (i = 12, 3, …, n) are collinear.

Definition 2.2. We say that three non-collinear points P, Q, R of pencil of straight lines defines the triple apex PQR of that pencil of straight lines. Straight lines PQ, PR, QR are the sides of the triple apex and points P, Q, R are apexes of the triple apex. We assume, for apex P, that it is opposite side of QR, and analogously for apexes Q and R.

Definition 2.3. Straight line which is defined with apex of the triple apex and with point of its opposite side is designated apex straight line 1. It will be designated with t_1 or $(QR)_1$. If it is obvious that the apex straight line 1 is dealt with, then 1 will be omitted.

Definition 2.4. The set of all points, all possible apex straight lines of the triple apex PQR represents plane PQR. We assume that two points P and Q are points of pencil of straight lines (O_∞) if their supports p and q are elements of that pencil of straight lines. As points of pencil of straight lines are defined like orderly set of (n-1) collinear points of planes E^2, the equality of points are defined like theequality of k sets. We can note following theorems:

Theorem 2.1. Any two different points P and Q of pencil of straight lines (O_∞) define exactly one straight line of that pencil of straight lines.

Proof. We can distinguish two cases. First, when supports of points P and Q are different straight lines and second when support is the same line. In the first case every pair of points P^i, Q^i in plane E^2 defines exactly one straight line and straight line PQ is closely defined. If p=q, then we have one straight line PQ = P^iQ^i = p = q.

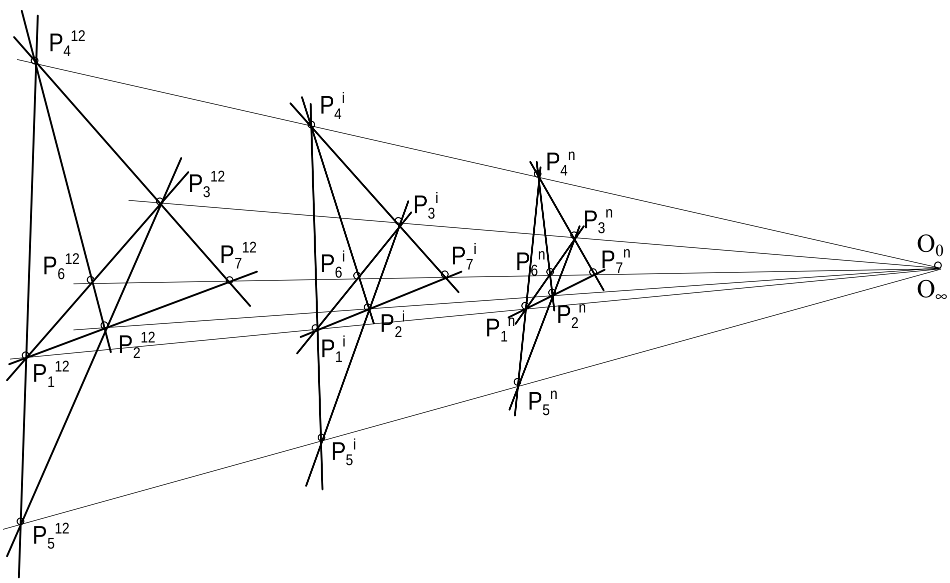

Fig. 1

If we supplement the plane E^2 with the infinitely distant straight line o_∞ , the P^2 real projective plane is obtained. We can, in a similar way, supplement the whole space E^n, whose plane is E^2, and obtain the projective space P^n. The center of pencil of straight lines (O_∞) is a point of plane P^2. We will prove the next theorem:

Theorem 2.2. If P_1, P_2, P_3, P_4 are four coplanar points of pencil of straight lines (O_∞), and not all of them are collinear, then the intersection of straight lines P_2P_3 and P_1P_4 implies the intersection of straight lines P_1P_3 and P_2P_4, and P_1P_2 and P_3P_4, too.

Proof. Let $P_1P_4 \cap P_2P_3 = P_5$, and let also $P^i_1P^i_4 \cap P^i_2P^i_3 = P^i_5$ ($i = 12, 3, \ldots, n$).To prove that straight lines P_1P_3 and P_2P_4 have intersection (also P_1P_2 and P_3P_4), we will consider points $P^i_6 = P^i_1P^i_3 \cap P^i_2P^i_4$ ($i = 12, 3, \ldots, n$).

As P^2 is a projective plane, there are points P^i_6. We should prove that support p_6, of point P_6 belongs to the considered pencil of straight lines (O_∞).

The real projective plane is a Dezargues' plane and for that reason perspectivity of the triple apexes $P_1^{12}P_3^{12}P_4^{12}$ and $P_1^iP_3^iP_4^i$ from point O_∞ produces their perspectivity from plane $p_{123} = p_{123}((P_1^{12}P_3^{12} \cap P_1^iP_3^i)(P_1^{12}P_4^{12} \cap P_1^iP_4^i))$. The triple apexes $P_3^{12}P_4^{12}P_5^{12}$ and $P_3^iP_4^iP_5^i$ are perspective, too. As $P_1^{12}P_4^{12} = P_4^{12}P_5^{12}$ and, also $(P_4^{12}P_5^{12} \cap P_4^nP_5^n)(P_3^{12}P_4^{12} \cap P_3^nP_4^n) = p_{345} = p_{134}$, we can conclude that the triple apexes $P_1^{12}P_3^{12}P_4^{12}$, $P_1^iP_3^iP_4^i$ and $P_3^{12}P_4^{12}P_5^{12}$, $P_3^iP_4^iP_5^i$ are perspective from the same point and the same straight line. The triple apexes $P_2^{12}P_3^{12}P_4^{12}$ and $P_2^iP_3^iP_4^i$ are perspective from the same point and from the same straight line. From relations:

$$(P_2^{12}P_4^{12} \cap P_2^iP_4^i) \in p_{134}$$
$$(P_1^{12}P_3^{12} \cap P_1^iP_3^i) \in p_{134},$$

we can conclude that the triple apexes $P_1^{12}P_4^{12}P_6^{12}$ and $P_1^iP_4^iP_6^i$ ($i = 3, 4, \ldots, n$) are perspective from the straight line p_{134}, which implies perspectivity from the point. As the $P_1^{12}P_1^i \cap P_4^{12}P_4^i = O_\infty$, the points P_6^{12}, P_6^i, P_6^n, O_∞ are collinear, which was the main purpose.

In the same way we may demonstrate that the straight lines P_1P_2 and P_3P_4 intersect each other in point P_7. The set of all points $X = X(X^{12},X^3,\ldots,X^n)$, the set of all straight lines defined with those points, the set of all planes defined with those points, the set of all objects that can be defined with notion of point, straight line, plane, and the set of all properties of those objects, we will designate a model of projective geometry M^n, which is defined on the pencil of straight lines (O_∞).

Not any two straight lines intersect each other in this kind of a model. It is a consequence of the multidimensionality of the model (dimension $n>2$). If the dimension of the model is $n=2$, than the model is reduced to a projective plane in which it is defined, and any two straight lines intersect each other.

Definition 2.5. The complete quadruple apex is a set of four coplanar points P_1, P_2, P_3, P_4, where none of the three of them are collinear, and six straight lines defined with those points. We call points - apexes, and straight lines - sides of the complete quadruple apex. The sides which do not belong to the same apex are opposite. Points

$$P_1P_2 \cap P_3P_4 = P_{12,34}\ ;\ P_1P_3 \cap P_2P_4 = P_{13,24}\ ;\ P_1P_4 \cap P_2P_3 = P_{14,23}$$

are diagonal points of the complete quadruple apex.

Definition 2.6. The complete quadruple apex is called F the quadruple apex if diagonal points of that quadruple apex are not collinear.

If diagonal points of some of the quadruple apexes are collinear, we call that quadruple apex, the AF the quadruple apex.

A plane is called an F plane if all of its quadruple apexes are the F quadruple apexes, and also, we call some plane AF plane if all of its quadruple apexes are the AF quadruple apexes.

Some plane is FAF plane if it contains the F and AF the quadruple apexes.

We will prove the following theorem:

Theorem 2.3. If pencil of straight lines (O_∞) belongs to Dezargous' and F plane, then every quadruple apex of the model M^n is an F quadruple apex.

Proof. If, contrary to the statement, there is a AF the quadruple apex $P_1P_2P_3P_4$, that would mean that points $P_{12,34}$, $P_{13,24}$, $P_{14,23}$, are collinear, which implies collinearity of points $P^i_{12,34}$, $P^i_{13,24}$, $P^i_{14,23}$. Collinearity of that points is in contrast with presumption that considered plane is a F plane.

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GEOMETRIJA PRAMENA PRAVIH

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U radu je posmatran pramen pravih u Euklidskoj, a zatim u projektivnoj ravni u kome je definisan model projektivne geometrije M^n čije su tačke skupovi kolinearnih $(n-1)$-torki čije prave-nosači pripadaju razmatranom pramenu.